

# Trajectory Planning Using High-Order Polynomials under Acceleration Constraint

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## Abstract

The trajectory planning, known as a movement from starting point to ending point by satisfying the constraints along the path, is an essential part of robot motion planning. A common way to create trajectories is to deal with polynomials which have independent coefficients. This paper presents a trajectory formulation as well as a procedure to arrange the suitable trajectories for applications. Created trajectories are aimed to be used for safe and smooth navigation in mobile robots. First, a trajectory problem is formulated by considering a border on the robot's acceleration as the constraint. Also, initial and final conditions for the robot's velocity along the straight path are settled. To investigate the idea that suggested trajectories perform motions with continuous velocity and smooth acceleration, three trajectory problems are formulated using 3<sup>rd</sup>, 4<sup>th</sup>, and 5<sup>th</sup> degrees of polynomials. The high-degree polynomials are used because of providing of smoothness, but there is complexity in the calculation of additional coefficients. To reduce the complexity of finding the high-degree polynomial coefficients, the acceleration constraint is simplified and this process is based on certain scenarios. Afterwards, the coefficients of the used polynomials are found by taking into account the acceleration constraint and velocity conditions. Additionally, to compare the obtained solutions through proposed scenarios, the polynomials' coefficients are solved numerically by Genetic Algorithm (GA). The computer simulation of motions, as well as acceleration constraint, shows that the velocity conditions at the beginning and at the end of motion are fulfilled.

*Keywords:* Motion planning, Trajectory planning, High-order polynomials, Velocity conditions, Acceleration constraint, Genetic algorithm

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## 1. Introduction

Trajectory planning is related to determining the robot's position, velocity, and acceleration during the motion time. However, when a mobile robot should traverse along a given path, there are infinite possible trajectories that the robot can run. Although only finite numbers of them are appropriate to run in applications. This paper focuses on trajectory planning for point-to-point motion by considering velocity and/or acceleration constraints at the initial and final points, as well as along the path. The trajectory planning taken in hand in here is similar to the vehicle trajectory in optimal control theory. Assume that a car moves through a linear or circular path. The problem here is related to how driver presses accelerator pedal to minimize the total driving time and to maximize the total travelling distance. In such kind of control problems, there exist constraints related to speed and acceleration limits. The trajectory planning taken in hand is similar to the car example, where we used a high-order polynomial as mathematical expression of travelling time and distance in the role of objective function under the velocity and acceleration constraints.

In the robotic studies, trajectory planning is investigated in many research papers. Elnagar et al. (2000) studied smooth piece-wise trajectories considering acceleration constraint. Choi et al. (2001) studied the near-time-optimal trajectory considering battery voltage and obstacle avoidance. Lepetic et al. (2003) studied the time minimizing in the spline curve path. Nguyen et al. (2007) studied polynomial s-curve trajectories. Haddad et al. (2007) focused on trajectories with limited velocities, accelerations, and torques as well as obstacle avoidance. Kardos et al. (2009) considered the trajectories composed of straight-line, circular segments, and continuous-curvature segments. Boryga et al. (2009) planned the trajectory in the form of higher-degree polynomials for serial-link robot manipulators. Korayem et al. (2013) formulated the trajectory based on the dynamic potential function model.

Jond et al. (2014) presented polynomial trajectories from third-order in closed-form solutions. Minh et al. (2014) proposed three trajectory generation techniques including flatness, polynomial, and symmetric polynomial trajectories subject to the vehicle constraints, and

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$$q(t) = -\frac{\Phi}{3t_f}t^3 + \frac{\Phi}{2}t^2, t \in [0 \quad t_f] \quad (10)$$

As it is seen, the 3<sup>rd</sup> degree polynomial solution is obtained. However, dealing with the higher-degree polynomials is difficult, because the number of possible scenarios increases exponentially.

### 2.2. 4<sup>th</sup> -degree Polynomial Trajectory

The optimization problem for 4<sup>th</sup> degree polynomial can be defined as follows:

$$\max_{\lambda_1, \lambda_2, \lambda_3} q(t) = \lambda_1 t^4 + \lambda_2 t^3 + \lambda_3 t^2 \quad (11)$$

$$s. t. \quad 4\lambda_1 t_f^3 + 3\lambda_2 t_f^2 + 2\lambda_3 = 0 \quad (12)$$

$$|12\lambda_1 t^2 + 6\lambda_2 t + 2\lambda_3| \leq \Phi \quad (13)$$

Here, the problem is complicated and the solution cannot be found by classical methods. To handle the problem, we propose an approach. The main idea is to simplify the acceleration constraint in Eq. (13). By this approach, we would only expect to find the near-optimal solutions.

Firstly, the nonlinear Eq. (13) can be reduced to two linear inequalities. In other words, Eq. (13) can be reordered at  $t=0$  and  $t=t_f$  as below.

$$|2\lambda_3| \leq \Phi, |12\lambda_1 t_f^2 + 6\lambda_2 t_f + 2\lambda_3| \leq \Phi \quad (14)$$

Afterwards, we consider Eq. (14) instead of Eq. (13). Here, we can choose a number of values for the right-hand side of Eq. (14) in order to convert the inequalities to equalities. Each chosen value establishes a scenario, and each scenario leads to finding a solution. However,

the number of values which can be chosen for the right-hand side of Eq. (14) is infinite; they can be considered among the fractional values within the interval  $[-\Phi, +\Phi]$ . Table 1 shows the problem solutions for some chosen values. Each solution gives a trajectory. Based on the maximum traveled distance generated by each trajectory during a given time, the suitable trajectories and also the suitable scenarios can be distinguished.

It is seen from Table 1 that the scenario (profile) number 2 is more suitable among the other scenarios. The scenario number 1 leads to losing one order of the polynomial. Among the rest of scenarios, the one that generates the maximum traveled distance can be a suitable choice for 4<sup>th</sup> degree polynomial trajectory. By the scenario number 2, the acceleration must be arranged at times  $t=0$  and  $t=t_f$  to  $0.5\Phi$  and  $-\Phi$ , respectively. Afterwards, the acceleration nonlinear constraint can be replaced with two linear inequalities that are given as follows:

$$\ddot{q}(0) \leq \frac{\Phi}{2} \Rightarrow 2\lambda_3 \leq \frac{\Phi}{2} \quad (15)$$

$$\ddot{q}(t_f) \leq -\Phi \Rightarrow 12\lambda_1 t_f^2 + 6\lambda_2 t_f + 2\lambda_3 \leq -\Phi \quad (16)$$

Solving the system containing Eq. (15) and Eq. (16) in equality condition, we can obtain:

$$\lambda_1 = -\frac{\Phi}{8t_f^2}, \lambda_2 = 0, \lambda_3 = \frac{\Phi}{4} \quad (17)$$

Therefore, a suitable 4<sup>th</sup> degree polynomial trajectory is obtained as follows:

$$q(t) = -\frac{\Phi}{8t_f^2}t^4 + \frac{\Phi}{4}t^2, t \in [0 \quad t_f] \quad (18)$$

Table 1  
Some possible scenarios with corresponding solutions for the 4<sup>th</sup> degree polynomial trajectory

Scn. No.	$\ddot{q}(0)$	$\ddot{q}(t_f)$	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\max q(t_f)$
1	$\Phi$	$-\Phi$	zero	$-0.333(\Phi/t_f)$	$0.5\Phi$	$0.166\Phi t_f^2$
2	$0.5\Phi$	$-\Phi$	$-0.125(\Phi/t_f^2)$	zero	$0.25\Phi$	$0.125\Phi t_f^2$
3	zero	$-\Phi$	$-0.25(\Phi/t_f^2)$	$0.333(\Phi/t_f)$	zero	$0.083\Phi t_f^2$
4	$-0.5\Phi$	$-\Phi$	$-0.375(\Phi/t_f^2)$	$0.666(\Phi/t_f)$	$-0.25\Phi$	$0.041\Phi t_f^2$
5	$-\Phi$	$-\Phi$	$-0.5(\Phi/t_f^2)$	$\Phi/t_f$	$-0.5\Phi$	zero

### 2.3. 5<sup>th</sup> -degree Polynomial Trajectory

The optimization problem for 5<sup>th</sup> degree polynomial is formulated as follows:

$$\max_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} q(t) = \lambda_1 t^5 + \lambda_2 t^4 + \lambda_3 t^3 + \lambda_4 t^2 \quad (19)$$

$$s. t. \quad 5\lambda_1 t_f^4 + 4\lambda_2 t_f^3 + 3\lambda_3 t_f^2 + 2\lambda_4 = 0 \quad (20)$$

$$|20\lambda_1 t^3 + 12\lambda_2 t^2 + 6\lambda_3 t + 2\lambda_4| \leq \Phi \quad (21)$$

Reordering the constraint given in Eq. (24) at  $t=0$ ,  $t=t_c$ , and  $t=t_f$

$$|2\lambda_4| \leq \Phi, |20\lambda_1 t_c^3 + 12\lambda_2 t_c^2 + 6\lambda_3 t_c + 2\lambda_4| \leq \Phi, |20\lambda_1 t_f^3 + 12\lambda_2 t_f^2 + 6\lambda_3 t_f + 2\lambda_4| \leq \Phi \quad (22)$$

where  $t_c$  is the critical point of the acceleration function. It

is well-known that a function reaches its maximums or minimums at the critical points. Therefore, when the acceleration constraint inequality is held at the critical point(s) as well as at  $t=0$  and  $t=t_f$ , then it will be held in all instances of interval  $[0, t_f]$ . Table 2 shows the problem solutions for 15 chosen states for the mentioned times ( $t=0$ ,  $t=t_c$ , and  $t=t_f$ ). It is seen from the table that the scenario number 6 is a suitable case. Using this scenario, the acceleration must be arranged at times  $t=0$  and  $t=t_f$  to  $0.5\Phi$  and  $-\Phi$ , respectively. Also,  $t_c$  is arranged to  $\frac{3t_f}{4}$ . Therefore, Eq. (21) can be replaced with two linear inequalities as well as a linear equality as below.

$$\ddot{q}(0) \leq \frac{\Phi}{2} \Rightarrow 2\lambda_4 \leq \frac{\Phi}{2} \quad (23)$$



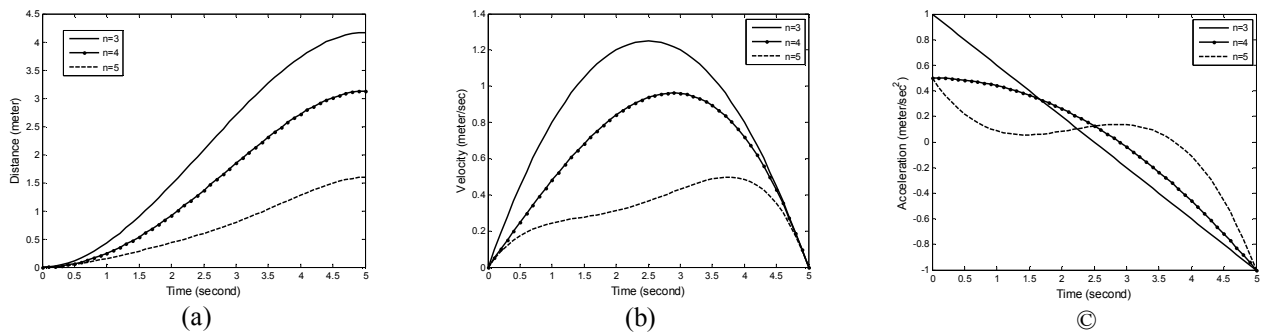


Fig. 1. Plots of the trajectories for, (a) position, (b) velocity and (c) acceleration

Table 3  
Scenario-based solutions versus GA optimal solutions

n	Scenario Based Solutions					GA Solutions				
	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$	max $q(t_f)$	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$	max $q(t_f)$
3	-0.0666	0.5000	-	-	4.1666	-0.0667	0.5005	-	-	4.1792
4	-0.0050	zero	0.2500	-	3.1250	-0.0042	-0.0104	0.2893	-	3.2969
5	-0.0029	0.0316	-0.1222	0.2500	1.5972	No feasible solution found				

#### 4. Conclusion

This study proposes a practical formulation for trajectory planning problem denoted by 3<sup>rd</sup>, 4<sup>th</sup>, and 5<sup>th</sup> degrees of polynomials. The formulation ensures that during the generated trajectories, the robot moves according to a bounded acceleration in order to have a safe navigation. The formulation also is taken into consideration of the velocity conditions. In this research, an approach based on scenarios is presented to find the coefficients of these polynomials. By simulating the trajectories by means of the obtained solutions, the consistency of the proposed scenarios is also investigated. The graphs of velocity and acceleration (shown in Fig. 1) of the generated trajectories show that resulting polynomials satisfy considered constraints and conditions. Additionally, suitability of the obtained solutions against the numerical optimal solutions of GA is clearly observed in this work (in Table 3).

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