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Presentation and Solving Non-Linear Quad-LevelProgramming Problem Utilizing a Heuristic Approach Based on Taylor Theorem

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Abstract

The multi-level programming problems are attractive for many researchers because of their application in several areas such as economic, traffic, finance, management, transportation, information technology, engineering and so on. It has been proven that even the general bi-level programming problem is an NP-hard problem, so the multi-level problems are practical and complicated problems therefore solving these problems would be significant. The literature shows several algorithms to solve different forms of the bi-level programming problems (BLPP).Not only there is no any algorithm for solving quad-level programming problem, but also it has not been studied by any researcher. The most important part of this paper is presentation and studying of a new model of non-linear multi-level problems. Then we attempt to develop an effective approach based on Taylor theorem for solving the non-linear quad-level programming problem. In this approach, by using a proposed smoothing method the quad-level programming problem is converted to a linear single problem. Finally, the single level problem is solved using the algorithm based on Taylor algorithm. The presented approach achieves an efficient and feasible solution in an appropriate time which has been evaluated by solving test problems.

Keywords: Non-Linearquad-level programming problem, Smoothing method, Taylor algorithm.

1. Introduction

It has been proven that the bi-level programming problem (BLPP) is an NP-Hard problem (J.F. Bard, 1991; L. Vicente, 1994). Several algorithms have been proposed to solve BLPP (G. Wang, 2010; N. V. Thoai, 2002; S.R. Hejazi, 2002; J. Yan, Xuyong.L, 2013;Wan, Z, L, 2014; Zheng, Y, 2014; Zhang , G, 2010; E. Hosseini, I.Nakhai Kamalabadi, 2013;J.F. Bard, 1998, 1991;Xu, P, & L. Wang, 2014; P. Xu, L. Wang, 2014) These algorithms are divided into the following classes: global techniques (Y. Jiang, X. Li, 2014; X. He, C. Li, T. Huang, 2014; Z. Wan, L. Mao, 2014), these algorithms obtain global optimal solution independently from characteristics such as initial solution and features of objective function. But the local methods are dependent to these characteristics. These methods is very complicated even for BLPP and we cannot use them for tri-level and quad-level. Enumeration methods (J. Nocedal, 2005;A.AL Khayyal, 1985), these methods calculate bounds of the objective function and try to meet feasible vertex points same as simplex method. In fact, the main concept is to achieve all of the feasible vertex points for BLPP and the best solution among them. Complexity is a challenge in these algorithms Transformation methods (Lv. Yibing, Hu. Tiesong, Wang, 2007; G. B. Allende, 2012), in these kinds of approaches the second level of the problem has been transformed by smooth methods, such as KKT conditions, to convert the problem into a single level problem. Then the obtained problem solved utilizing non-linear methods.Metaheuristic approaches (R. Mathieu, 1994;T. X. Hu, Guo, 2010; B. Baran Pal, 2010; Z. G.Wan, 2012; E. Hosseini, & I.Nakhai Kamalabadi, 2013, 2014, 2015, 2017; Y. Zheng, 2014), these algorithms have been interested by many different researchers to solve optimization problems in general and BLPP particularly. Here inspired algorithm has been proposed which searches randomly in the feasible region. These methods are very fact, the challenge is that they are approximate approaches and propose a solution near the optimal solution. Fuzzy methods (M. Sakava, I. Nishizaki, Y. Uemura, 1997; S Sinha, 2003; S. Pramanik, 2009; S.R. Arora, 2007), these approaches using membership functions for constraints and objective functions. In fact, the problem will be simplified using membership functions. Primal-dual interior methods (G. Z. Wang, 2008). In the following, these techniques are shortly introduced.

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However there are many approaches to solve the BLPP and this model of multi-level has been studied by many researchers, but there is no any attempt for modeling and presentation of the quad-level programming problem (QLPP). In this paper, the authors have tried to propose a new model of multi-level programming problem, QLPP, and then it will be solved using the proposed method. Finally a new heuristic approach is proposed which is based on Taylor method.
All of pervious proposed algorithms have been applied to solve BLPP and for multi-level, particularly quad-level, programming problems aren't used. In fact, quad-level is a new model of multi-level programming problems which is proposed at the first time in this paper and it needs a novel algorithm too. The proposed algorithm in this paper has three parts in general. At the first, the follower levels (second, third and fourth levels) are smoothed utilizing mathematical theorems and the quad-level programming problem will be converted in this part of the algorithm where some non-linear constraints are appeared. Then the method uses Taylor theorem to approximate the non-liner constraints and to convert them to linear. Finally, the linear single-level obtained problem will be solved using enumeration method. In fact all feasible vertices are checked and the best one is introduced as an optimal solution.

The remainder of the paper is structured as follows: problem formulation and smooth method to the QLPP are introduced in Section 2.
The algorithm based on analytic theorems and Taylor theoremis proposed in Section 3. Computational results are presented for our approach in the Section 4. As result, the paper is finished in Section 5 by presenting the concluding remarks.

2. Problem Formulation

2.1. The linear bi-level and tri-level programming problems

In this section models of bi-level and tri-level programming problems are introduced. BLPP is used frequently by problems with decentralized planning structure. It is defined as (J.F. Bard, 1991):

$$\begin{aligned} &\min_x F(x,y) \\ &s.t \min_y f(x,y) \\ &s.t\ g(x,y) \le 0, \\ &x,y \ge 0. \end{aligned} \tag{1}$$

Where

$F: R^{n\times m} \dot{\to} R^1, f: R^{n\times m} \dot{\to} R^1,$
$g: R^{n\times m} \dot{\to} R^q, x \in R^n, y \in R^m$

In general, BLPP is a non-convex optimization problem; therefore, there is no general algorithm to solve it. This problem can be non-convex even when all functions and constraints are bounded and continuous. A summary of important properties for convex problem are as follows (J. Nocedal, S.J. Wright, 2005; A.AL Khayyal, 1985), which $F: S \dot{\to} R^n$ and S is a nonempty convex set in R^n:

(1) The convex function f is continuous on the interior of S.
(2) Every local optimal solution of F over a convex set $X \subseteq S$ is the unique global optimal solution.
(3) If $\nabla F(\bar{x}) = 0$, then $\bar{x}$ is the unique global optimal solution of Fover S.

Because a tri-level decision reflects the principle features of multi-level programming problems, the algorithms developed for tri-level decisions can be easily extended to multi-level programming problems which the number of levels is more than three. Hence, just tri-level programming is studied in this paper.
In a TLPP, each decision entity at one level has its objective and its variables in part controlled by entities at other levels. To describe a TLPP, a basic model can be written as follows (E. Hosseini, I.Nakhai Kamalabadi, 2015):

$$\begin{aligned} &\min_x F_1(x,y,z) \\ &\quad s.t \min_y F_2(x,y,z) \\ &\quad\quad s.t \min_z F_3(x,y,z) \\ &s.t\ \ g(x,y,z) \le 0, \\ &\quad\quad\quad x,y,z \ge 0. \end{aligned} \tag{2}$$

2.2. The non-linear quad-level programming problems
We propose the QLPP model as following formulation:

$$\begin{aligned} &\min_x F_1(x,y,z,t) \\ &\quad s.t \min_y F_2(x,y,z,t) \\ &\quad\quad s.t \min_z F_3(x,y,z,t) \\ &\quad\quad\quad s.t \min_t F_4(x,y,z,t) \end{aligned} \tag{3}$$

$$\begin{aligned} &s.t\ \ g(x,y,z,t) \le 0, \\ &\quad\quad\quad x,y,z,t \ge 0. \end{aligned} \tag{4}$$

Where $x \in R^n, y \in R^l, z \in R^p, t \in R^s$, and the variables x, y, z, t are called the top-level, second-level, third-level, and bottom-level variables respectively, $F_1(x,y,z,t), F_2(x,y,z,t), F_3(x,y,z,t), F_4(x,y,z,t)$, the top-level, second-level, third-level, and bottom-level objective functions, respectively. In this problem each level has individual control variables, but also takes account of other levels' variables in its optimization function.

2.3. Smooth method for QLPP

2.3.1. Definition

Every point such as (x, y, z, t) is a feasible solution to tri-level problem if $(x, y, z, t) \in S$

Definition 2.3.2:

Every point such as (x^*, y^*, z^*, t^*) is an optimal solution to the tri-level problem if

$F(x^*. y^*, z^*, t^*) \leq F(x, y, z, t) \ \forall \ (x, y, z, t) \in S.$
Using KKT conditions for the last levels in problem (3), the following problem is constructed:

$$\begin{aligned}
&\min_x F_1(x, y, z, t)\\
&\quad \min_y F_2(x, y, z, t)\\
&\quad \min_{z,t} F_2(x, y, z, t)\\
&\qquad s.t\ \nabla_t L(x, y, z, t, \mu) = 0,\\
&\qquad \mu g(x, y, z, t) = 0, \qquad (5)\\
&\qquad g(x, y, z, t) \leq 0,\\
&\qquad x, y, z, t, \mu \geq 0.
\end{aligned}$$

Where L is the Lagrange function and $L(x, y, z, t, \mu) = F_4(x, y, z, t) + \mu g(x, y, z, t)$.KKT conditions have been used for both last levels in problem (5), therefore the following problem is obtained:

$$\begin{aligned}
&\min F_1(x, y, z, t)\\
&\quad s.t\ \ \nabla_y N(x, y, z, t, \mu, \alpha, \beta, p_1, p_2, p_3, p_4) = 0,\\
&\quad p\, \nabla_z K(x, y, z, t, \mu, \alpha, \beta) = 0,\\
&\quad q\ \alpha\ \nabla_t L(x, y, z, t, \mu) = 0,\\
&\quad v\ \beta\ \mu g(x, y, z, t) = 0, \qquad (6)\\
&\quad w g(x, y, z, t) = 0,\\
&\quad g(x, y, z, t) \leq 0,\\
&\quad x, y, z, t, \alpha, \beta, \mu, p_1, p_2, p_3, p_4 \geq 0.
\end{aligned}$$

Where K and N are the Lagrange functions and $K(x, y, z, t, \mu, \alpha, \beta) = F_3(x, y, z, t) + \mu g(x, y, z, t) + \alpha\ \nabla_z L(x, y, z, t, \mu) + \beta\ \mu g(x, y, z, t)$and

$$\begin{aligned}
N(x, y, z, t, \mu, \alpha, \beta, p_1, p_2, p_3, p_4) &= F_2(x, y, z, t) + \ p_1 \nabla_z K(x, y, z, t, \mu, \alpha, \beta)\\
&+ p_2\ \alpha\ \nabla_t L(x, y, z, t, \mu)\\
&+ p_3\ \beta\ \mu g(x, y, z, t) + \ p_4 g(x, y, z, t)
\end{aligned}$$

Now we use the following point to convert the most complicated constraint,$p_3\ \beta\ \mu g(x, y, z, t) = 0$, to three simpler constraints: If $ab = ac = bc = 0$, then $abc = 0$.

By applying above point for problem (6), two times, and $a = \beta,\ b = \mu,\ c = g(x, y, z)$,the following problem is obtained:

$$\begin{aligned}
&\min F_1(x, y, z, t)\\
&\quad s.t\ \ \nabla_y N(x, y, z, t, \mu, \alpha, \beta, p_1, p_2, p_3, p_4) = 0,\\
&p_1 \nabla_z K(x, y, z, t, \mu, \alpha, \beta) = 0,\\
&p_2 \nabla_t L(x, y, z, t, \mu) = 0,\\
&\alpha\ \nabla_t L(x, y, z, t, \mu) = 0, p_2\ \alpha\ = 0,\\
&p_3 \beta\ = 0,\\
&p_3\ \mu = 0,\\
&p_3\ g(x, y, z, t) = 0, \qquad (7)\\
&\beta\ \mu = 0,\\
&\beta\ g(x, y, z, t) = 0,\\
&\mu g(x, y, z, t) = 0,\\
&p_4 g(x, y, z, t) = 0,\\
&g(x, y, z, t) \leq 0,\\
&x, y, z, t, \alpha, \beta, \mu, p_1, p_2, p_3, p_4 \geq 0.
\end{aligned}$$

Because problem (7) has a complementary constraint, it is not convex and it is not differentiable. In this paper we propose new functions for smoothing complementary constraints in problem (7). Using the following smooth method, problem (7) is smoothed, and then the final problem is solved using an algorithm based on Taylor theorem.
If $m \geq 0,\ n \geq 0$, Let, $\phi: R^2 \to R$, $\phi(m, n) = 2m - n - \sqrt{4m^2 + n^2}$, , then we have: $\phi(m, n) = 0 \Leftrightarrow 2m - n - \sqrt{4m^2 + n^2} = 0 \Leftrightarrow 2m - n = \sqrt{4m^2 + n^2} \Leftrightarrow (2m - n)^2 = 4m^2 + n^2 \Leftrightarrow 4m^2 + n^2 - 4mn = 4m^2 + n^2 \Leftrightarrow -4mn = 0 \Leftrightarrow mn = 0.$

Now let $\phi: R^3 \to R,\ \phi(m, n, \mathcal{E}) = 2m - n - \sqrt{4m^2 + n^2 - \mathcal{E}}$, then we have: $\phi(m, n, \mathcal{E}) = 0 \Leftrightarrow 2m - n - \sqrt{4m^2 + n^2 - \mathcal{E}} = 0 \Leftrightarrow 2m - n = \sqrt{4m^2 + n^2 - \mathcal{E}} \Leftrightarrow (2m - n)^2 = 4m^2 + n^2 - \mathcal{E} \Leftrightarrow 4m^2 + n^2 - 4mn = 4m^2 + n^2 - \mathcal{E} \Leftrightarrow -4mn = -\mathcal{E} \Leftrightarrow mn = \frac{\mathcal{E}}{4}, m \geq 0, n \geq 0.$

Using the proposed function $\phi(m, n, \mathcal{E}) = 2m - n - \sqrt{m^2 + n^2 - \mathcal{E}}$ in problem (7), we obtain the following problem:
$\min F_1(x, y, z, t)$

$$s.t \quad 2p - \nabla_z K(x,y,z,t,\mu,\alpha,\beta) - \sqrt{4p^2 + \nabla_z K(x,y,z,t,\mu,\alpha,\beta)^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2q - \nabla_t L(x,y,z,t,\mu) - \sqrt{4q^2 + \nabla_t L(x,y,z,t,\mu)^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2\alpha - \nabla_t L(x,y,z,t,\mu) - \sqrt{4\alpha^2 + \nabla_t L(x,y,z,t,\mu)^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2q - \alpha - \sqrt{4q^2 + \alpha^2 - \varepsilon} = \frac{\varepsilon}{4}, \qquad (8)$$

$$2v - \beta - \sqrt{4v^2 + \beta^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2v - \mu - \sqrt{4v^2 + \mu^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2v - g(x,y,z,t) - \sqrt{4v^2 + g(x,y,z,t)^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2\beta - \mu - \sqrt{4\beta^2 + \mu^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2\beta - g(x,y,z,t) - \sqrt{4\beta^2 + g(x,y,z,t)^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2\mu - g(x,y,z,t) - \sqrt{4\mu^2 + g(x,y,z,t)^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$2w - g(x,y,z,t) - \sqrt{4w^2 + g(x,y,z,t)^2 - \varepsilon} = \frac{\varepsilon}{4},$$

$$\nabla_y N(x,y,z,t,\mu,\alpha,\beta,p,p_2,p_3,p_4) = 0,$$

$$g(x,y,z,t) \le 0,$$

$$x,y,z,t,\alpha,\beta,\mu,p,q,v,w \ge 0.$$

Which in the constraints$m = \alpha,\beta,p,q\ ,v,w \ge 0, n = \nabla_z K(x,y,z,t,\mu,\alpha,\beta), \nabla_t L(x,y,z,t,\mu), -g(x,y,z,t)$.

LetH_1, H_2, H_3 as follows for three first constraints:

$H_1(x,y,z,t,p_i)$=

$$\begin{bmatrix} 2p_1 - \mathrm{g}_1(\mathrm{x,y,z,t}) - \sqrt{{p_1}^2 + \nabla_z K(x,y,z,t,\mu,\alpha,\beta)_1^2 - \varepsilon} \\ 2p_2 - \mathrm{g}_2(\mathrm{x,y,z,t}) - \sqrt{{p_2}^2 + \nabla_z K(x,y,z,t,\mu,\alpha,\beta)_2^2 - \varepsilon} \\ \vdots \\ 2p_l - \mathrm{g}_l(\mathrm{x,y,z,t}) - \sqrt{{p_l}^2 + \nabla_z K(x,y,z,t,\mu,\alpha,\beta)_l^2 - \varepsilon} \end{bmatrix} \qquad (9)$$

$H_2(x,y,z,t,q_i)$=

$$\begin{bmatrix} 2q_1 - \nabla_t L(x,y,z,t,\mu)_1 - \sqrt{{q_1}^2 + \nabla_t L(x,y,z,t,\mu)_1^2 - \varepsilon} \\ 2q_2 - \nabla_t L(x,y,z,t,\mu)_2 - \sqrt{{q_2}^2 + \nabla_t L(x,y,z,t,\mu)_2^2 - \varepsilon} \\ \vdots \\ 2q_l - \nabla_t L(x,y,z,t,\mu)_l - \sqrt{{q_l}^2 + \nabla_t L(x,y,z,t,\mu)_l^2 - \varepsilon} \end{bmatrix} \qquad (10)$$

H_3
(x,y,z,t,α_i)=

$$\begin{bmatrix} 2\alpha_1 - \nabla_t L(x,y,z,t,\mu)_1 - \sqrt{{\alpha_1}^2 + \nabla_t L(x,y,z,t,\mu)_1^2 - \varepsilon} \\ 2\alpha_2 - \nabla_t L(x,y,z,t,\mu)_2 - \sqrt{{\alpha_2}^2 + \nabla_t L(x,y,z,t,\mu)_2^2 - \varepsilon} \\ \vdots \\ 2\alpha_l - \nabla_t L(x,y,z,t,\mu)_l - \sqrt{{\alpha_l}^2 + \nabla_t L(x,y,z,t,\mu)_l^2 - \varepsilon} \end{bmatrix} \qquad (11)$$

Also we define H_j, $j = 4,5,\ldots,11$similar above for 4-th to 11-th constraints.

$$H_i'(x,y,z,t,\alpha) = H(x,y,z,t,\alpha) - \frac{\varepsilon}{4}\ , \qquad (12)$$

Problem (7) can be written as follows:

$$\min F_1(\mathrm{x,y,z,t})$$

$$s.t \quad H_j'(x,y,z,t,\alpha_i) = 0, \quad \mathrm{j} = 1,2,\ldots,11, \quad \mathrm{i} = 1,2,\ldots,\mathrm{l}.$$

$$\nabla_y N(x,y,z,t,\mu,\alpha,\beta,p,p_2,p_3,p_4) = 0, \qquad (13)$$

$$g(x,y,z,t) \le 0,$$

$$x,y,z,t,\alpha,\beta,\mu,p,q,v,w \ge 0.$$

Where $T = (\mathrm{x,y,z,t}) \in R^{k+l+p+s}$

Because problem (8) equal to (13), we use the following method for solving problem (13).

3. The proposed algorithm based onTaylor method (TA)

Definition 3.1: A metric space is pair (X,d) where X is a set and d is a metric on X and:

(i) $d \ge 0$,
(ii) $d(x,y) = 0 \Leftrightarrow x = y$,
(iii) $d(x,y) = d(y,x)$,
(iv) $d(x,y) \le d(x,z) + d(z,y)$.

Definition 3.2: A sequence $\{x_n\}$ is said to Cauchy if for every $\varepsilon > 0$ there is an N such that

$$\forall_{m>r>N} |x_m - x_r| < \varepsilon.$$

Theorem 3.1 (Taylor Theorem)(A. Silverman. Richard, 2000):Suppose f has n +1 continuous derivatives on an open interval containing a. Then for each
x in the interval,

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{k}(a)}{k!}(x-a)^{k}\right] + R_{n+1}(x) \quad (14)$$

Where the error term $R_{n+1}(x)$, for some c between a and x, satisfies

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad (15)$$

This form for the error $R_{n+1}(x)$, is called the Lagrange formula for the reminder.

The infinite Taylor series converge to f,

$$f(x) = \left[\sum_{k=0}^{\infty} \frac{f^{k}(a)}{k!}(x-a)^{k}\right]$$

If only if$\lim_{n\to\infty} R_{n+1}(x) = 0$. (16)

Proof:

The proof of this theorem was given by (A. Silverman. Richard, 2000).Taylor Theorem is a great tool for linearize the non-linear functions which are continuous and differentiable. This theorem is very applicable in engineering and practical problems to approximate complicated functions to polynomials.

It is clear to see that functions H_i, $i = 1,2,\dots,11$ in (13) are always continuous everywhere. Therefore it is possible to use Taylor Theorem for them. By applying the theorem 4.1 to a feasible point such as T^k for functionsH_i, $i = 1,2,\dots,11$, and taking only two linear parts of them in problem (16), the following linear functions are constructed: ForH'_j:

$$H'_{ji}(T^k,\alpha) + \nabla H'_{ji}(T^k,\alpha)(T - T^k) = 0, \quad i = 1,2,\dots,s, \quad j = 1,2,\dots,11. \quad (17)$$

Let

$$P_j(T) = \begin{bmatrix} P_{j1}(T) \\ P_{j2}(T) \\ \vdots \\ P_{js}(T) \end{bmatrix} = \begin{bmatrix} H'_{j1}(T^k,\alpha) + \nabla H'_{j1}(T^k,\alpha)(T-T^k) \\ H'_{j2}(T^k,\alpha) + \nabla H'_{j2}(T^k,\alpha)(T-T^k) \\ \vdots \\ H'_{js}(T^k,\alpha) + \nabla H'_{js}(T^k,\alpha)(T-T^k) \end{bmatrix} \quad j = 1,2,\dots,11 \quad (18)$$

WhichH'_{j1}, is i-th component inH'_j.The obtained problem using Taylor theorem is a linear programming and it can be solved using linear algorithm such as simplex method.

The steps of the proposed algorithm are as follows:

Step 1: Initialization

The feasible point T^k is created randomly, error$\mathcal{E}_1$ is given and we suppose that $k = 1$, $F(T) = F_1(x,y,z,t)$, $\mathcal{E}_1$ is a small and appropriate given error and finishing the algorithm depends on $\mathcal{E}_1$ such that it is finished whenever the difference between produced solutions by the algorithm in two consecutive iterations is less than $\mathcal{E}_1$.

Step 2: Finding solution

Using Taylor theorem for $H'_j, j = 1,2,\dots,11$at T^k and (17), in problem (13) we obtain the following problem:

$$\begin{aligned} &\min F_1(x,y,z,t) \\ &s.t \quad P_j(T) = 0, \quad j = 1,2,\dots,11, \\ &\quad \nabla_y N(x,y,z,t,\mu,\alpha,\beta,p,p_2,p_3,p_4) = 0, \quad (19) \\ &\quad g(x,y,z,t) \le 0, \\ &\quad x,y,z,t,\alpha,\beta,\mu,p,q,v,w \ge 0. \end{aligned}$$

Step 3: Making the present best solution

Because (19) is an approximation for (13) by Taylor theorem, therefore, the optimal solution for (19) is an approximation of the optimal solution for (13). Thus T^{k+1} can be a good approximation of optimal solution problem (13). Therefore, we let $T^* = T^{k+1}$ and go to the next step.

Step 4: Termination

If $d(F(T^{k+1}), F(T^k)) < \mathcal{E}_1$ then the algorithm is finished andT^* is the best solution by the proposed algorithm. Otherwise, let k=k+1 and go to the step 2. Which d is metric and,

$$d\left(F(T^{k+1}), F(T^k)\right) = \left(\sum_{i=1}^{n+p+s+l} (F(T_i^{k+1}) - F(T_i^k))^2\right)^{\frac{1}{2}}.$$

Following theorems show that the proposed algorithm is convergent.

Theorem 3.2: Every Cauchy sequence in real line and complex plan is convergent.

Proof:

Proof of this theorem is given in [34].

Theorem 3.3: Sequence $\{F_k\}$ which was proposed in above algorithm is convergent to the optimal solution, so that the algorithm is convergent.

Proof:

Let

$(F_v) = (F(t^v)) =$
$(F(t_1^v), F(t_2^v), \dots, F(t_{n+2m}^v))= (F_1^{(v)}, F_2^{(v)}, \dots, F_{n+p+s+l}^{(v)})$.

According to step 4

$$d(F_{k+1},F_k)=\mathrm{d}\left(\mathrm{F}(\mathrm{T}^{k+1}),\mathrm{F}(\mathrm{T}^{k})\right)=(\sum_{i=1}^{n+p+s+l}(F(T_i^{k+1})-F(T_i^{k}))^2)^{\frac{1}{2}}<\varepsilon_1 \qquad (20)$$

therefore $(\sum_{i=1}^{n+p+s+l}\left(F(T_i^{k+1})-F(T_i^{k})\right)^2)<{\varepsilon_1}^2$

. There is large number such as N which k+1>k>N and j=1,2,…, $n+p+s+l$ we have:

$(F_j^{(k+1)}-F_j^{(k)})^2<{\varepsilon_1}^2$, therefore $\left|F_j^{(k+1)}-F_j^{(k)}\right|<\varepsilon_1$

$$Now\ let\ m=k+1, r=k\ then\ we\ have$$

$$\forall_{m>r>N}\left|F_j^{(m)}-F_j^{(r)}\right|<\varepsilon_1.$$

This shows that for each fixed j, $(1\le \mathrm{j}\le n+p+s+l)$, the sequence $(\mathrm{F}_\mathrm{j}^{(1)},\mathrm{F}_\mathrm{j}^{(2)},\dots)$ is Cauchy of real numbers, then it converges by theorem 4.5.

Say, $F_j^{(m)}\to F_j$ as $m\to\infty$. Using these $n+p+s+l$ limits, we define $F=(F_1,F_2,\dots,F_{n+p+s+l})$.From (17) and m=k+1, r=k,

$$d(F_m,F_r)<\varepsilon_1$$

Now if $\mathrm{r}\to\infty$, by$F_r\to$ F we have $d(F_m,F)\le\varepsilon_1$.
This shows that F is the limit of (F_m) and the sequence is convergent by definition 3.3 therefore proof of theorem is finished.

Theorem 3.4: If sequence $\{\mathrm{f}(\mathrm{t_k})\}$ is converge to f(t) and f be linear function then $\{\mathrm{t_k}\}$ is converge to t.

Proof:

Proof of this theorem is given in [30].

Theorem 3.5: Problems (13) and (16) are equal therefore they have same optimal solutions.

Proof:

It is sufficient to prove that,$\left|H_j'(\mathrm{T})-P_j(\mathrm{T})\right|<\varepsilon$,j=1,2,…,11 for every arbitrary ε>0. According to the theorem 4.4 and (19), (20) we have:

$P_j(\mathrm{T})=H_j'(\mathrm{T^k})+\nabla H_j'(\mathrm{T^k})(\mathrm{T}-\mathrm{T^k})$

H_j' (T)

$=H_j'(\mathrm{T^k})+\nabla H_j'(\mathrm{T^k})(\mathrm{T}-\mathrm{T^k})+\nabla^2 H_j'(\mathrm{T^k})\frac{(\mathrm{T}-\mathrm{T^k})^2}{2}+R_n(T).$

$$\left|H_j'(T)-P_j(T)\right|=\left|\nabla^2 H_j'(\mathrm{T^k})\frac{(\mathrm{T}-\mathrm{T^k})^2}{2}+R_n(T)\right|$$

$$\le\left|\nabla^2 H_j'(\mathrm{T^k})\frac{(\mathrm{T}-\mathrm{T^k})^2}{2}\right|+|R_n(T)|$$

Now if $n\to\infty$, from (18) $|R_n(T)|<\frac{\varepsilon}{2}$ and let $\left|\nabla^2 H_j'(\mathrm{T^k})\right|<m$ that m is an arbitrary large number, this is possible because $\nabla^2 H_j'(\mathrm{T^k})$ is a number.

If $k\to\infty$ because F is linear then by theorems 4.6 and 4.7 $\mathrm{T^k}\to$ Ttherefore $\left|\mathrm{T^k}-\mathrm{T}\right|<\varepsilon_2$, say $\varepsilon_2=\sqrt{\frac{\varepsilon}{m}}$

$$\Rightarrow\left|H_j'(T)-P_j(T)\right|\le\left|\nabla^2 H_j'(\mathrm{T^k})\frac{(\mathrm{T}-\mathrm{T^k})^2}{2}\right|+|R_n(T)|$$

$$\le\left|\nabla^2 H_j'(\mathrm{T^k})\right|\left|\frac{(\mathrm{T}-\mathrm{T^k})^2}{2}\right|+|R_n(T)|$$

$$\le m.\frac{\varepsilon}{2m}+\frac{\varepsilon}{2}=\varepsilon.$$

4. Computational Results

There are several practical problems which can be modeled as a quad-level programming problems. One of these problems is supply – chain which has been mentioned here.The supply-chain has four levels in decision: the first level is customs, the second level is products importer, the third level is products wholesaler and the last level is products badger. The decision maker at all four levels try to maximize their own benefits as their objective functions, and each has its own constraints and variables. The importer considers the decision making process of the customs, the wholesaler considers the decision-making process of the importer, and the badger considers the decision-making process of the wholesaler. At the same time, the customs decisions take into account the reaction of the importer, the importer's decisions take into account the reaction of the wholesaler, and the wholesaler likewise takes the reaction of the badger into account. The importer wants to maximize own profits and the wholesaler likes to maximize his (her) benefits and the badger wants to maximize own objective function. This problem can be established by a linear quad-level programming model to obtain the optimal solution to determine the cost and price.

To illustrate the algorithm, three examples will be solved using the algorithm.

Example 1:

The following QLPP will be solved by the proposed algorithm.

$\min_{\mathrm{x}} x^2+4\mathrm{y}-2\mathrm{z}+t$

s.t

$\min_{\mathrm{y}} 7x-y^2+21z-2t$

s.t

$\min_{\mathrm{z}} -x+7y+z^2-t^2$

s.t

$\min_{\mathrm{t}} -x+3y+2xz-3t^2$

s.t

$\mathrm{x}-3\mathrm{y}+z^2+\mathrm{t}\le 32,$

$-3x + 5yz - z - t \leq 101,$
$3x^2 + 5y - z + 2t \leq 168,$
$x, y, z, t \geq 0.$

The problem has been solved using the proposed algorithm and the best solutions have been shown in the Table 1 and Table 2. Behavior of the variables has been shown in figure 1.

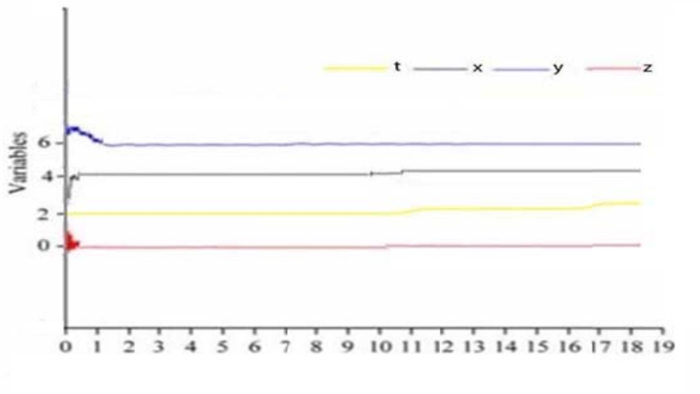

Fig. 1.Behavior of the variables for $\varepsilon = 0.001$ - Example 1

Table 1
Objective functions in the best solution by the proposed algorithm – Example 1

Optimal	Best solution by our method	Iterations	Time
(x^*, y^*, z^*, t^*)	(7.01,4.00,5.22,2.53)	4000	2.58s
$F_1(x^*, y^*, z^*, t^*)$	57.23		
$F_2(x^*, y^*, z^*, t^*)$	137.63		
$F_3(x^*, y^*, z^*, t^*)$	41.82		
$F_4(x^*, y^*, z^*, t^*)$	58.97		

Table 2
Different solutions in different iterations – Example 1

Optimal Solution	Best solution by our method	Iterations	Time
(x^*, y^*, z^*, t^*)	(7.01,4.00,5.22,2.53)	400	2.58s
(x^*, y^*, z^*, t^*)	(4.03,5.87,0.00,2.01)	4000	3.41s
(x^*, y^*, z^*, t^*)	(4.39,6.00,0.03,2.83)	19000	5.17s

Example 2:
Consider the following problem:

$$\max_x \; x + 4y^2 + 2xz + t$$
s.t
$$\max_y \; xz + y + z + tx$$
s.t
$$\max_z \; xy^2 - 2y + 2z^2 - ty$$
s.t
$$\max_z \; xy - y + 3z + tz$$
s.t
$$-x - y \leq -3,$$
$$3xy^2 - 2y + z + t \leq 10,$$
$$-2x + y - 2z - t \leq -1,$$
$$x, y, z, t \geq 0.$$

The problem has been solved using the proposed algorithm and we present the best solutions in the Table 3 and Table 4. Behavior of the variables has been shown in figure 2.

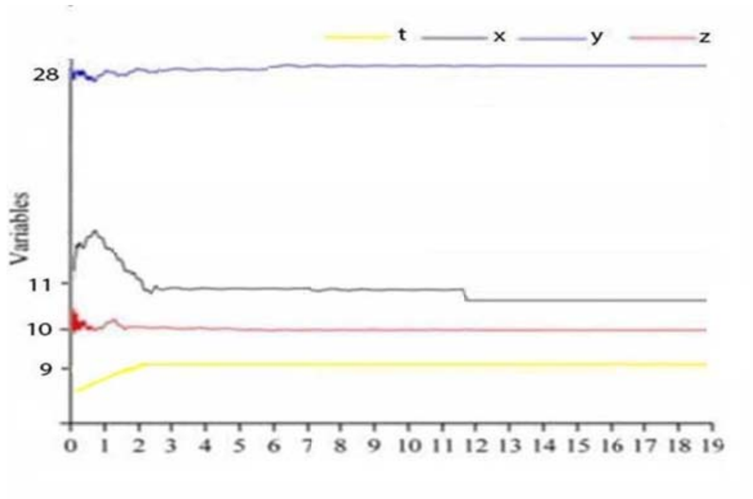

Fig. 2.Behavior of the variables for $\varepsilon = 0.001$ - Example 2

Table 3
Objective functions in the best solution by the proposed algorithm – Example 2

Optimal Solution	Best solution by our method	Iterations	Time
(x^*, y^*, z^*, t^*)	(0.0,5.51,5.02,4.38)		
$F_1(x^*, y^*, z^*, t^*)$	125.82		
$F_2(x^*, y^*, z^*, t^*)$	10.53	500	2.24s
$F_3(x^*, y^*, z^*, t^*)$	15.24		
$F_4(x^*, y^*, z^*, t^*)$	31.53		

Table 4
Different solutions in different iterations – Example 2

Optimal Solution	Best solution by our method	Iterations	Time
(x^*, y^*, z^*, t^*)	(12.09,28.12,10.23,8.37)	1000	2.56s
(x^*, y^*, z^*, t^*)	(11.21,28.57,10.09,8.55)	10000	3.35s
(x^*, y^*, z^*, t^*)	(10.83,28.92,10.01,9.02)	19000	4.53s

Example 3:

Consider the following non-linear quad-level programming problem:

$\min_{x} x^2 + y^2 - 3z^2 + t$

$\min_{y} x - 4y + 2z + 4t$

s.t

$-x - y - 2t \leq -3,$

$-3x + 2y - z + 2t \geq -10,$

$\min_{z} x + y - z$

s.t

$-2x + y - 2z \leq -1,$

$2x + y + 4z - t \leq 14,$

$\min_{t} x - 2y - 2z - t$

s.t

$2x - y - z + t \leq 2,$

$x, y, z \geq 0.$

Using KKT conditions, the following problem is obtained:

$$\min_{x} x^2 + y^2 - 3z^2 + t$$

$$\begin{aligned}
&\text{s.t}\\
&-x - y - 2t \le -3,\\
&3x - 2y + z + 2t \le 10,\\
&-2x + y - 2z \le -1,\\
&2x + y + 4z - t \le 14,\\
&\beta_1(-2x + y - 2z + 1) = 0,\\
&\beta_2(2x + y + 4z - t - 14) = 0,\\
&\beta_1 + \beta_2 = 1,\\
&2x - y - z + t \le 2,\\
&\mu(2x - y - z + t - 2) = 0,\\
&\mu(-1) = -2,\\
&x, y, z, t, \beta_1, \beta_2, \mu \ge 0.
\end{aligned}$$

By the proposed smooth method, the above problem will be converted to:

$$\min_{x} x^2 + y^2 - 3z^2 + t$$

$$\begin{aligned}
&\text{s.t}\\
&-x - y - 2t \le -3,\\
&3x - 2y + z + 2t \le 10,\\
&2\beta_1 - (-2x + y - 2z + 1) - \sqrt{{\beta_1}^2 + (-2x + y - 2z + 1)^2 + \varepsilon} = 0,\\
&2\beta_2 - (2x + y + 4z - t - 14) - \sqrt{{\beta_2}^2 + (2x + y + 4z - t - 14)^2 + \varepsilon} = 0,\\
&2\mu - (2x - y - z + t - 2) - \sqrt{\mu^2 + (2x - y - z + t - 2)^2 + \varepsilon} = 0,\\
&\beta_1 + \beta_2 = 1,\\
&\mu(-1) = -2,\\
&x, y, z, t, \beta_1, \beta_2, \mu \ge 0.
\end{aligned}$$

Now using Taylor theorem, non-linear constraints in the above single – level problem are approximated to the simpler constraints. Finally, this problem is infeasible after solving the problem by the proposed method.

5. Conclusion and Future Work

In this paper, a new model of non-linear multi-level programming problem which has four levels was been proposed. This model has not been studied already by any researcher. Also a new heuristic approach has been presented to convert the non-linear quad-level problem into a single level problem. Then, using an algorithm based on Taylor theorem linear approximation single problem was been obtained. Utilizing the proposed mathematics analyze theorems the optimal solution was proposed. Our algorithm has acceptable numerical results and present good solutions. In the future works, the following should be researched:

(1) Examples in larger sizes can be supplied to illustrate the efficiency of the proposed algorithm.

(2) Showing the efficiency of the proposed algorithms for solving other kinds of QLPP such as quadratic and non-linear QLPP.

Nomanclature

$F_1(x,y,z,t)$	Objective function of the first level in the QLPP
$F_2(x,y,z,t)$	Objective function of the second level in the QLPP
$F_3(x,y,z,t)$	Objective function of the third level in the QLPP
$F_4(x,y,z,t)$	Objective function of the fourth level in the QLPP
u	Slack variable
v	Slack variable
w	Slack variable
$F(x,y)$	Objective function of the first level in the BLPP
$f(x,y)$	Objective function of the first level in the BLPP
g(x, y)	Constraints in the BLPP
S	Feasible region of the QLPP
IR	Inducible region of the QLPP
α	The last feasible values of u
β	The last feasible values of v
μ	The last feasible values of w
ε	An arbitrary very small positive number
(x^*, y^*, z^*, t^*)	Optimal solution for the QLPP
(x^*, y^*, z^*)	Optimal solution for the TLPP
(x^*, y^*)	Optimal solution for the BLPP

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