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Sensitivity Analysis of Simple Additive Weighting Method (SAW): The Results of Change in the Weight of One Attribute on the Final Ranking of Alternatives

Azizollah Memariani[a], Abbas Amini[b], Alireza Alinezhad[c,*]

[a]*Department of mathematics & computer sciences, school of economic sciences, Tehran, Iran.*

[b] *Department of industrial engineering, Tarbiat Modares University, Tehran, Iran.*

[c] *Department of industrial engineering, Islamic Azad University, Qazvin Branch, Qazvin, Iran.*

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Abstract

Most of data in a multi-attribute decision making (MADM) problem are unstable and changeable, then sensitivity analysis after problem solving can effectively contribute to making accurate decisions. This paper provides a new method for sensitivity analysis of MADM problems so that by using it and changing the weights of attributes, one can determine changes in the final results of a decision making problem. This analysis applied for SAW technique, one of the most used multi-attribute decision making techniques, and the formulas are obtained.

Keywords: Multi-attribute decision making (MADM); SAW Technique; Sensitivity analysis; Ranking methods; Attribute weight.

1. Introduction

Multi-attribute decision making models are selector models and used for evaluating, ranking and selecting the most appropriate alternative among alternatives. Alternatives in an MADM problem are evaluated by k attributes and the most appropriate alternative is selected or they are ranked in accordance with attribute's value for each alternative and the importance of each attribute for decision maker. MADM model is formulated as a decision making matrix as follow:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_k \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1k} \\ d_{21} & d_{22} & \dots & d_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \dots & d_{mk} \end{bmatrix} \end{array}$$

In this matrix $A_1,A_2,A_3,\dots A_m$ are available and predetermined m alternatives and $C_1,C_2,C_3,\dots,C_k$ are Effective k attributes in decision making that are used for measuring utility of each alternative and d_{ij} are special value of attribute j[th] for alternative i[th], in other words the efficiency of the alternative i[th] against the attribute j[th].

The most important issue in MADM models is that, the data used in them are unstable and changeable, so, sensitivity analysis after problem solving can effectively contribute to making accurate decisions. Because the weights are acquired from the opinions of decision maker (DM), so DM wants to know that which attribute is more sensitive than others and how much change in the weight of one attribute can change the final results of the solved problem.

Sensitivity analysis for MADM models is one of discussed issues in MADM field and many researches have done at last decades about it. The first researches in this field are the works of [2], [3] and [8] that focused on determining decision sensitivity to probabilistic estimation errors. [9] And [1] suggested a sensitivity analysis for additive MADM models. They assumed a set of weights for attributes and obtained a new set of weights for them, so that the efficiency of alternatives has been equal or the order of them has changed. In the research of [5] the structure of weights' set was studied

*corresponding author E-mail: alinezhad@qiau.ac.ir

and conditions that lead to special ranking or priority of one alternative to another, in additive decision making models, were discussed. [6] by offering a method at the frame of algorithms in sensitivity analysis, have studied the result of changes in attributes' weights on the final score of alternatives in MADM models and calculate the required change in attributes' weights for changing the optimal solution. These algorithms and methods were revised by [7]. [11] Studied two types of sensitivity analysis for two methods of MADM. First, they determined the most sensitive attribute and calculated the change in attributes weights that leads to change in the ranking of alternatives and second type measures the sensitivity of decision making matrix elements [12] proposed a model for determining method sensitivity to changes of separate parameters enables to increase the reliability of the applied methods. [10] Studied sensitivity analysis approach for produce complementary information by determination of criteria values domain in decision making matrix.

In this paper we offer a new method for sensitivity analysis of multi-attribute decision making problems so that by using it and changing the weights of attributes, one can determine changes in the results of a decision making problem. This analysis has done for SAW technique and the formulas are obtained.

The reminder of paper is organized as follow: in section 2, SAW technique is reviewed and formulas and relations are mentioned. Section 3 is the most important part of the paper and proposes a new method for sensitivity analysis of MADM models. Section 4, presents a numerical example and examines the accuracy of proposed model. Finally, in section 5, conclusions and suggestions for future researches are cited.

2. Review on SAW Technique

SAW Technique is one of the most used MADM techniques. It is simple and is the basis of most MADM techniques such as AHP and PROMETHEE that benefits from additive property for calculating final score of alternatives. In SAW technique, final score of each alternative is calculated as follow and they are ranked.

$$P_i = \sum_{j=1}^{k} w_j . r_{ij} \quad ; \quad i = 1, 2, \ldots, m \tag{1}$$

Where r_{ij} are normalized values of decision matrix elements and calculated as follow:
For profit attributes, we have:

$$r_{ij} = \frac{d_{ij}}{d_j^{Max}} \quad ; \quad d_j^{Max} = \underset{1 \le i \le m}{Max}\, d_{ij} \quad ; \quad j = 1, 2, \ldots, k \tag{2}$$

And for cost attributes

$$r_{ij} = \frac{d_j^{Min}}{d_{ij}} \quad ; \quad d_j^{Min} = \underset{1 \le i \le m}{Min}\, d_{ij} \quad ; \quad j = 1, 2, \ldots, k \tag{3}$$

If there is any qualitative attribute, then we can use some methods for transforming qualitative variables to quantitative ones [4].

3. Developing new method for sensitivity analysis of MADM problems

In classic techniques of MADM, often, it is assumed that all used data (such as weight of attributes, efficiency of alternatives against attributes,…) are deterministic then final score or utility of alternatives are obtained by solving MADM, whereas in reality, data of decision making problem are changing. So that, after solving decision making problems, usually a sensitivity analysis must be done for them.

In former researches were done about sensitivity analysis of MADM problems, often focused on determining the most sensitive attribute so that the least change at it, change the current ranking of alternatives, also focused on finding this least value of change. But, a new method for sensitivity analysis of MADM problems is considered in this paper that calculates the changing in the final score of alternatives when a change occurs in the weight of one attribute.

3.1. The effect of change in the weight of one attribute on the weight of other attributes

The vector for weights of attributes is $W^t = (w_1, w_2, ..., w_k)$ wherein weights are normalized and sum of them is 1, that is:

$$\sum_{j=1}^{k} w_j = 1 \tag{4}$$

With these assumptions, if the weight of one attribute changes, then the weight of other attributes change accordingly, and the new vector of weights transformed into

$$W'^t = (w'_1, w'_2, ..., w'_k) \tag{5}$$

The next theorem depicts changes in the weight of attributes.

Theorem 3-1-1: In the MADM model, if the weight of attribute P[th], changes as Δ_p, then the weight of other attributes change as Δ_j; j=1,2,…,k .

$$\Delta_j = \frac{\Delta_p \cdot w_j}{w_p - 1} \quad ; j = 1,2,\dots,k \quad , \quad j \neq p \tag{6}$$

Proof: If new weights of attributes are w'_j and new weight of attribute P^th^ changes as

$$w'_p = w_p + \Delta_p \tag{7}$$

Then new weight of other attributes would change as

$$w'_j = w_j + \Delta_j ;\; j = 1,2,\dots,k \,,\, j \neq p \tag{8}$$

And because the sum of weights must be 1 then

$$\sum_{j=1}^{k} w'_j = \sum_{j=1}^{k} w_j + \sum_{j=1}^{k} \Delta_j \;\Rightarrow\; \sum_{j=1}^{k} \Delta_j = 0 \tag{9}$$

Therefore

$$\Delta_p = -\sum_{\substack{j=1 \\ j \neq p}}^{k} \Delta_j \tag{10}$$

Wherein

$$\Delta_j = \frac{\Delta_p \cdot w_j}{w_p - 1} ;\; j = 1,2,\dots,k, j \neq p \tag{11}$$

Since

$$\begin{aligned} -\Delta_p &= \sum_{\substack{j=1 \\ j \neq p}}^{k} \Delta_j = \sum_{\substack{j=1 \\ j \neq p}}^{k} \frac{\Delta_p \cdot w_j}{w_p - 1} \\ &= \frac{\Delta_p}{w_p - 1} \sum_{\substack{j=1 \\ j \neq p}}^{k} w_j \\ &= \frac{\Delta_p}{w_p - 1}(1 - w_p) = -\Delta_p \end{aligned} \tag{12}$$

Discussion: In an MADM problem, if the weight of attribute P^{th} changes from w_p to w'_p as:

$$w'_p = w_p + \Delta_p \tag{13}$$

Then, the weight of other attributes would change as

$$w'_j = \frac{1 - w_p - \Delta_p}{1 - w_p} \cdot w_j = \frac{1 - w'_p}{1 - w_p} \cdot w_j \quad ; j = 1,2,\dots,k \quad , j \neq p \tag{14}$$

Since, for $j = 1,2,..,k \quad , j \neq p$ we have

$$\begin{aligned} w'_j &= w_j + \Delta_j = w_j + \frac{\Delta_p \cdot w_j}{w_p - 1} \\ &= \frac{w_j (w_p - 1) + \Delta_p \cdot w_j}{w_p - 1} \Rightarrow \\ w'_j &= \frac{(1 - w_p - \Delta_p) \cdot w_j}{1 - w_p} \\ &= \frac{1 - w'_p}{1 - w_p} \cdot w_j \;;\; j = 1,2,\dots,k, j \neq p \end{aligned} \tag{15}$$

Then, new vector for weights of attributes would be $W'^t = (w'_1, w'_2, \dots, w'_k)$, that

$$w'_j = \begin{cases} w_j + \Delta_p & if \quad j = p \\ \dfrac{1 - w'_p}{1 - w_p} . w_j & if \quad j \neq p, \; j = 1,2,\dots,k \end{cases} \tag{16}$$

$$\begin{aligned} & w'_p = w_p + \Delta_p \\ & \Rightarrow \begin{cases} if \quad w'_p > w_p & \Rightarrow w'_j < w_j \\ if \quad w'_p < w_p & \Rightarrow w'_j > w_j \end{cases} \\ & \qquad j = 1,2,\dots,k \quad , j \neq p \end{aligned} \tag{17}$$

The sum of new weights of attributes in (16) is 1, since:

$$\begin{aligned} \sum_{j=1}^{k} w'_j &= \sum_{\substack{j=1 \\ j \neq p}}^{k} w'_j + w'_p \\ &= \sum_{\substack{j=1 \\ j \neq p}}^{k} \frac{w_j (1 - w_p - \Delta_p)}{1 - w_p} + w_p + \Delta_p \\ &= \frac{(1 - w_p - \Delta_p)}{1 - w_p} \sum_{\substack{j=1 \\ j \neq p}}^{k} w_j + w_p + \Delta_p \\ &= \frac{(1 - w_p - \Delta_p)}{1 - w_p} .(1 - w_p) + w_p + \Delta_p \\ &= 1 - w_p - \Delta_p + w_p + \Delta_p = 1 \end{aligned} \tag{18}$$

Corollary: In the new vector for weights that obtained from (16), the weight's ratio is constant (exception of attribute P^{th}) because new weights for attributes (exception of attribute P^{th}) is obtained by multiplying the constant value $\frac{(1 - w_p - \Delta_p)}{1 - w_p}$ to old weight of them, then, the ratio of new weight of attribute C_i to new weight of attribute C_j for $i, j = 1,..., k$, $i, j \neq p$ is equal to the ratio of old ones. That is

$$\frac{w'_i}{w'_j} = \frac{w_i}{w_j}; i, j = 1,..., k, i, j \neq p \qquad (19)$$

3.2. The effect of change in the weight of one attribute on the final score of alternatives in SAW Technique

In a decision making problem as SAW, if the weight of one attribute changes, then the final score of alternatives will change. The next theorem calculates this change.

Theorem 3-2-1: In the MADM model of SAW, if the weight of attribute P^th^ changes as Δ_p, then the final score of alternative i^th^, $i = 1,2,..., m$ would change as δ_i that is

$$\delta_i = \Delta_p \cdot r_{ip} + \sum_{\substack{j=1 \\ j \neq p}}^{k} \frac{\Delta_p \cdot w_j}{w_p - 1} \cdot r_{ij} , \quad i = 1,2,..., m \qquad (20)$$

Proof: If we define δ_i as the difference between the old and new score of alternative ith and if we consider the differences between new and old weights in (16), then for each alternative i, we have:

$$\delta_i = P'_i - P_i = \sum_{j=1}^{k} (w'_j - w_j) \cdot r_{ij}$$

$$= \Delta_p \cdot r_{ip} + \sum_{\substack{j=1 \\ j \neq p}}^{k} \frac{\Delta_p \cdot w_j}{w_p - 1} \cdot r_{ij} \qquad (21)$$

$$\mathrm{i} = 1,2,..., \mathrm{m}$$

By using (14), new score of alternatives are calculated as:

$$w'_j = \frac{1 - w'_2}{1 - w_2} \cdot w_j \ ; j = 1,3,4$$

$$\Rightarrow w'_j = 0.75 w_j \qquad (22)$$

$$\Rightarrow W'^t = (0.3, 0.4, 0.225, 0.075)$$

$$P'_i = \sum_{j=1}^{k} w'_j \cdot r_{ij}$$

$$= \sum_{\substack{j=1 \\ j \neq p}}^{k} \frac{(1 - w_p - \Delta_p) \cdot w_j}{1 - w_p} \cdot r_{ij} + (w_p + \Delta_p) \cdot r_{ip}$$

$$= (\frac{1 - w_p - \Delta_p}{1 - w_p}) \sum_{\substack{j=1 \\ j \neq p}}^{k} w_j \cdot r_{ij} + w_p \cdot r_{ip} + \Delta_p \cdot r_{ip}$$

$$= (\frac{1 - w_p - \Delta_p}{1 - w_p}) \sum_{j=1}^{k} w_j \cdot r_{ij} + \frac{\Delta_p}{1 - w_p} \cdot w_p \cdot r_{ip} + \Delta_p \cdot r_{ip}$$

$$, i = 1,2, ..., m$$

Then, new score of alternative i^th^ with regard to its old score and value of change in the weight of attribute P^th^, would be

$$P'_i = (1 - \frac{\Delta_p}{1 - w_p}) P_i + \frac{\Delta_p}{1 - w_p} \cdot r_{ip} \qquad (23)$$

$$i = 1,2, ..., m$$

From the above equation it is clear that new score of each alternative is calculated by considering its old score and value of change in the weight of attribute P^th^, so that, this property can simply been utilized in computer programming for calculating new score of one alternative by considering its old score and the value of change at the weight of one attribute.

4. Numerical example

We assume an MADM problem that has three alternatives and four attributes wherein attributes C_1, C_4 are of cost type and attributes C_2, C_3 are of profit type (the weights of attributes found out from the methods of Entropy, Eigen vector, Linmap or weighted least square which are suitable.

$W^t = (0.4, 0.2, 0.3, 0.1)$

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{r} A_1 \\ D = A_2 \\ A_3 \end{array} & \begin{bmatrix} 13 & 9 & 9 & 8 \\ 5 & 3 & 5 & 12 \\ 7 & 5 & 7 & 6 \end{bmatrix} \end{array}$$

For solving it by SAW technique, linearly normalized matrix, according to the relations in section 2, is:

$$R = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.38 & 1 & 1 & 0.75 \\ 1 & 0.33 & 0.56 & 0.50 \\ 0.71 & 0.56 & 0.78 & 1 \end{bmatrix} \end{array}$$

And final score of each alternative is calculated by

$$P_i = \sum_{j=1}^{k} w_j \cdot r_{ij} \qquad ,i = 1,2,....,m$$

That m=3 and k=4.

$P_1 = 0.72, \quad P_2 = 0.684, \quad P_3 = 0.73$

Therefore

$A_3 > A_1 > A_2$

Now we assume that the weight of 2nd attribute increased as $\Delta_2 = 0.2$ and is $w_2' = w_2 + \Delta_2 = 0.2 + 0.2 = 0.4$, then by (14), the weight of other attributes change as by considering this new vector for the weight of attributes, we resolve the problem and calculate the new score of alternatives by using matrix R as

$$P_i' = \sum_{j=1}^{k} w_j' \cdot r_{ij} \qquad ,i = 1,2,....,m$$

Wherein m=3 and k=4.

$P_1' = 0.819, \quad P_2' = 0.596, \quad P_3' = 0.747$

Therefore $A_1 > A_3 > A_2$, and it is clear that the ranking has changed from the old one.

If we use equation (23), without resolving the problem, we can calculate the final score of alternatives by technique SAW and changing in the weight of 2nd attribute as follow

$$P_i' = \left(\frac{1 - w_2'}{1 - w_2}\right) P_i + \frac{\Delta_2}{1 - w_2} \cdot r_{i2}$$
$$i = 1,2,3$$

$P'_1 = 0.795, \quad P'_2 = 0.596, \quad P'_3 = 0.687$ Therefore $A_1 > A_3 > A_2$. It is obvious that, this result is as same as the result of resolving the problem.

Corollary: Accordingly said at corollary of theorem (3-1-1), the ratio of new and old weights of all attributes except attribute 2, will not change, that is

$$\frac{w_i'}{w_j'} = \frac{w_i}{w_j} \qquad ; \; i, j = 1,3,4$$

For example, for attributes 1st and 4th we have

$$\frac{w_1'}{w_4'} = \frac{w_1}{w_4} \Rightarrow \frac{0.3}{0.075} = \frac{0.4}{0.1} = 4$$

This example demonstrates that: First, changing in the weight of one attribute affects the weight of other attributes and the amount of this change is calculated by (14). Second, the final score of all alternatives will change after this change, however, there is no need for resolving the problem and the change in the final score of alternatives is calculated by (23).

5. Conclusions and future research

Decision making is the integral part of human life. Regardless of the variety of decision making problems, we can categorize them into two categories, multi-objective decision making problems that decision maker must design an approach that has the most utility by considering limited resources and multi-attribute decision making problems that decision maker must select one alternative from among available alternatives so that has the most utility. Naturally, for selecting an alternative we must consider several and often conflicting attributes.

Generally, all MADM problems can be depicted as a matrix. Each row of this matrix is illustrative of one alternative and each column is illustrative of one attribute and its elements are the efficiency of alternatives against attributes. The attributes that are chosen for decision making are conflicting, usually. This means that, improvement at one attribute may result in the deflation of other attributes. Also, by regarding the relative importance of attributes, one can assign weight for them. By assuming a vector for the weights of attributes and elements of decision making matrix, MADM problems can be solved by available techniques and select the best alternative or rank them [4].

In classic techniques of MADM, often, it is assumed that all used data (such as weight of attributes, efficiency of alternatives against attributes,…) are deterministic then final score or utility of alternatives are obtained by solving MADM, whereas in reality, data of decision making problem are changing. So that, after solving decision making problems, usually a sensitivity analysis must be done for them.

The researches have done at sensitivity analysis for MADM problems, often focused on determining the most sensitive attribute in model. This attribute is one that, the least change in its weight relative to that of others, leads to change in ranking of alternatives. Also, they found the value of changing in the weight of one attribute that leads to change in ranking of alternatives. These studies frequently focused on attributes' sensitivity.

Other type of sensitivity analysis that is not addressed at former studies, is calculating the change in the final score of alternatives in light of changing in the weight of particular attribute. In this sensitivity analysis, for a given change in the weight of one attribute, the change in the score of alternatives is calculated.

This type of sensitivity analysis can be applied in MADM related software for solving decision making problems so by adding it to this software and by utilizing graphical capability of computer, one can change the weight of one attribute arbitrarily and observe its effect on the final score and rank of alternatives, immediately. Following suggestions are proposed for future researches:

- Studying the effect of changing in one element of decision making matrix on the final score of alternatives in SAW technique.
- Studying the effect of simultaneously changing in the weight of one attribute and in the one element of decision making matrix, on the final score of alternatives in SAW technique.
- Applying this type of sensitivity analysis for other techniques of MADM such as PROMETHEE and AHP.

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