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Research Article

A Fractile Model for Stochastic Interval Linear Programming Problems

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Abstract

In this paper, we first introduce a new category of mathematical programming where the problem coefficients are interval random variables. These problems include two different kinds of ambiguity in the problem coefficients which are being interval and being random. We use Fractile method to solve these problems. In this method, using the existing method, we change the interval problem coefficients to the random mode and then we solve the random problem using Fractile method. Also, a numerical example is presented to show the effectiveness of this model. Finally, we emphasize that this approach can be useful for the model with multi-objective as a generalized model in the future study.

Keywords: Random variable, Random interval variable, Random interval programming, Fractile model

1. Introduction

The probability theory is one of the basic principles of modern mathematics which is related to other fields of mathematics such as algebra, topology, analysis, geometry, dynamical systems and it is one of the most important means to describe the complexity of uncertainty of the parameters. Also, its close relationships and commonalities with other fields of study such as computer sciences, ergodic theory, cryptography, game theory, analysis, differential equations, mathematics and physics, economics and statistical mechanics (Knill, 2009) has caused this theory to be practical in different fields including economics (Hildenbrand (1975), random geometry (Matheron (1975) or to be used in confronting with vague and imprecise information.

In most decision makings, quantities used are not accurate data, but are dependent to the environmental conditions. Furthermore, collecting accurate information so that they are not dependent on the human diagnosis and judgment is very difficult or impossible in practice. This uncertainty could result from the incomplete, erroneous, missing or unknown data in different applications (Nasseri and Bavandi (2018)). This uncertainty sometimes could happen for a random variable which is called stochastic programming. Stochastic programming provides a framework for modeling the decision making problems which contain inaccurate data (S. H. Nasseri and S. Bavandi (2019), Bavandi, Nasseri and Triki (2020)). To formulate a stochastic programming problem, we should estimate a proper probability distribution which parameters obey. However, the estimation is not always a simple task because historical data of some parameters cannot be obtained easily especially when we face a new uncertain variable, and subjective probabilities cannot be specified easily when many parameters exist. Moreover, even if we succeeded to estimate the probability distribution from historical data, there is no guarantee that the current parameters obey the distribution actually. There are various approaches in the literature that can be used to solve the stochastic programming (Kall and Mayer (2004), Sakawa, Yano and Nishizaki (2013). For decision problems under probabilistic uncertainty, from a different viewpoint, Charnes and Cooper (1959) proposed chance constrained programming which admits random data variations and permits constraint violations up to specified probability limits. Also Charnes and Cooper (1963) considered three types of decision rules, the minimum or maximum expected value model, the minimum variance model, and the maximum probability model for optimizing objective functions with random variables, which are referred to as the expectation model, the variance model, and the probability model, respectively. Moreover, Kataoka (1963) and Geoffrion (1967) individually proposed the fractile model.

Sometimes, accurate measurement of the random data is impossible, therefore in these cases each random variable would be defined as an interval, so studying the Linear Programming models with Interval Coefficients (LPIC) (Suprajitno and Mohd (2008) will be considered. Many researchers investigated interval linear programming problems on the basis of order relations between two intervals (Chanas and Kuchta (1996), Inuiguchi and Kume (1994), Jana and Panda (2014), Sengupta, Pal and Chakraborty (2001). Interval linear programming problems have been studied by several authors, such as Bhurjee and Panda (2016), Ishibuchi and Tanaka (1989), Chanas and Kuchta (1996), Hladik (2015), Hladik (2014), Gen and Cheng (1997) and Wang and Jin (2019). For example, Ishibuchi and Tanaka

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(1989), studied linear programming problems where the objective function has interval coefficients and they transformed this problem into a standard objective optimization problem. Kruse and Meyer (1987) have developed a methodology to solve a nonlinear interval optimization problem by transforming that to a general optimization problem which is free from interval uncertainty. Nasseri and Bavandi (2017), considered a Stochastic Interval-Valued Linear Fractional Programming problem. In this problem, the coefficients and scalars in the objective function are fractional-interval, and technological coefficients and the quantities on the right side of the constraints were random variables with the specific distribution. Subulan (2020), proposed a novel interval programming and chance constrained optimization based hybrid solution approach for a fully uncertain, multi-objective and multi-mode resource investment project scheduling problem.

Random intervals are one of the categories of the random set with a wide range of applications in different industries and sciences and it is taken into consideration of many researchers in recent years. Miranda, Couso and Gil (2005) has presented one of the most important references for full introduction of the interval random variables. This paper considers a linear programming problem involving random interval coefficients. A random interval programming model is presented by extending the Fractile model of stochastic programming. The main problem includes interval random variables and using the proposed model, it is turned to a certain Equivalent.

This paper is organized as follows: In Section 2, we present some basic definitions of the intervals and the interval random variables which are required for our discussion. We also state some basic concepts and characteristics of the probability theory. In Section 3, we present Linear Programming with Interval Coefficients (LPIC). Also, we define optimistic optimal value and pessimistic optimal value. In Section 4, we present a model on the basic of the fractile model of stochastic programming for solving Linear Programming problem involving Random Interval Coefficients (LPRIC). In Section 5, provides an illustrative example and the corresponding results. Finally, Section 6 is devoted to concluding remarks.

2. Preliminaries

In this section, we recall some basic concepts of interval arithmetic, which is taken from (Moore, Kearfott and Clou (2009)).

2.1. The basic interval arithmetic

Definition 2.1. A closed real interval $[\underline{a},\bar{a}]$ denoted by A, is a real interval number which can be defined completely by

$$A=[\underline{a},\bar{a}]=\{\alpha:\underline{a}\le\alpha\le\bar{a},\alpha\in\mathbb{R}\}$$

Where $\underline{a}$ and $\bar{a}$ are the left and right limits of A, respectively.

Definition 2.2. An interval is called unbounded, if the lower bound or the upper bound are infinity.i.e. $(-\infty,4],[7,\infty),(-\infty,\infty)$ and etc.

Definition 2.3. Let $*\in\{+,-,\times,\div\}$ be a binary operation on $\mathbb{R}$. If A and B are arbitrary closed intervals, then

$$A*B=\{a*b:a\in A,b\in B\}$$

In case of division, it is assumed that $0\notin B$.

Let A be an interval and $r\in\mathbb{R}$ be a constant, then

$$\begin{cases} rA=r[\underline{a},\bar{a}]=[r\underline{a},r\bar{a}], & if \quad r\ge 0,\\ rA=r[\underline{a},\bar{a}]=[r\bar{a},r\underline{a}], & if \quad r<0. \end{cases}$$

2.2 Axiomatic Probability

The primary reference for sections 2.2 and 2.3 are (Casella and Berger (2001), Grimmett and Stirzaker (2001)). Probability theory is derived from a small set of axioms and a minimal set of essential assumptions. The first concept in probability theory is the sample space, which is an abstract concept containing primitive probability events.

Definition 2.4. The sample space is a set Ω that contains all possible outcomes.

Definition 2.5. An event ω is a subset of the sample space Ω.

An event may be any subsets of the sample space (including the entire sample space), and the set of all events is known as the event space.

Definition 2.6. The set of all events in the sample space Ω is called the event space and is denoted $\mathcal{F}$.

Assembling a sample space, event space and a probability measure into a set produce what is known as a probability space.

Definition 2.7. A probability space is denoted using the tuple $(\Omega,\mathcal{F},P)$ where Ω is the sample space, $\mathcal{F}$ is the event space and P is the probability set function which has the domain $\omega\in\mathcal{F}$.

2.3 Random variables

Studying the behaviour of random variables, and more importantly, functions of random variables (i.e. statistics) are essential. This section covers univariate random variables.

Definition 2.8. Let $(\Omega,\mathcal{F},P)$ be a probability space. If $X:\Omega\to\mathbb{R}$ is a real-valued function has as its domain elements of Ω, then X is called a random variable.

A random variable is essentially a function which takes $\omega\in\Omega$ as an input an produces a value $x\in\mathbb{R}$ where is the $\mathbb{R}$ symbol for the real line. Random variables come in one of three forms: discrete, continuous and mixed. Random variables which mix discrete and continuous distributions are generally less important in financial economics and so here the focus is on discrete and continuous random variables.

The set of all random variables in $\mathbb{R}$ will be shown by $I^S(\mathbb{R})$.

Definition 2.9. A random variable is called discrete if its range consists of a countable (possibly infinite) number of elements.

While discrete random variables are less useful than continuous random variables, they are still commonly encountered.

Discrete random variables are characterized by a Probability Mass Function (PMF) which gives the probability of observing a particular value of the random variable.

Definition 2.10. The probability mass function for a discrete random variable X is defined as $f(x)=P(x)$, for all $x\in R(X)$ and $f(x)=0$, for all $x\notin R(X)$, where $R(X)$ is the range of X (i.e. the values for which X is defined).

Definition 2.11. A random variable is called continuous if its range is uncountably infinite and there exists a non-negative-valued function $f(x)$ defined or all $x\in(-\infty,\infty)$ such that for any event $B\subset R(X)$, $P(B)=\int_{x\in B} f(x)dx$ and $f(x)=0$, for all $x\notin R(X)$, where $R(X)$ is the range of X (i.e. the values for which X is defined).

The PMF of a discrete random variable is replaced with the Probability Density Function (pdf) for continuous random variables.

Definition 2.12. For a continuous random variable, the function f is called the Probability Density Function (PDF). A function $f:\mathbb{R}\to\mathbb{R}$ is a member of the class of continuous density functions if and only if $f(x)\geq 0$ for all $x\in(-\infty,\infty)$ and $\int_{-\infty}^{\infty} f(x)dx=1$.

Definition 2.13. The Cumulative Distribution Function (CDF) of a random variable X is defined as $F(c)=P(x\leq c)$, for all $c\in(-\infty,\infty)$.

The cumulative distribution function is used for both discrete and continuous random variables. When X is a discrete random variable, the CDF is

$$F(x)=\sum_{s\leq x} f(s),$$

for $x\in(-\infty,\infty)$ and when X is a continuous random variable, the CDF is

$$F(x)=\int_{-\infty}^{x} f(s)ds,$$

for $x\in(-\infty,\infty)$.

2.4. Random interval variables

Generally, a random interval variable is a measurable function from a probability space to a collection of closed intervals. In other words, a random interval variable is a random variable which takes interval values. The following definition can be used for random interval variables which are given from Miranda, Couso and Gil (2005).

Definition 2.13. Given a probability space (Ω,E,P), $a(\omega)=\left[\underline{a}(\omega),\bar{a}(\omega)\right]$ is a random interval variable defined on Ω, if $\underline{a}(\omega),\bar{a}(\omega)$ are random variables, and for any $\omega\in\Omega$, $\underline{a}(\omega)\leq\bar{a}(\omega)$. In the other words, if (Ω,E,P) be a probability space, where $\Omega=\{\omega_1,\omega_2,...,\omega_m\}$ and $\mathrm{I}_1,\mathrm{I}_2,...,\mathrm{I}_m$ be intervals, then a function X with

$$X(\omega)=\begin{cases} I_1 & , \quad if \quad \omega=\omega_1 \\ I_2 & , \quad if \quad \omega=\omega_2 \\ \vdots & \qquad\qquad \vdots \\ I_m & , \quad if \quad \omega=\omega_m \end{cases}$$

is defined as a random interval variable.

3. Linear Programming with Interval Coefficients

Some or all coefficients are the interval in this linear programming and to solve it, numerous problems have to be optimized. Therefore, it cannot be solved by the classic programming methods and more suitable methods are needed. According to Shaocheng method (Shaocheng (1994)), to the interval programming problems we can divide the problem to two normal programming problems and get the optimized answer in an interval, one of the answers considers the best and the other considers the worst solution. Of course, there are some ways in which one normal programming problem is created instead of solving two, but we intended to consider the Shaocheng method for random linear programming, which will be discussed in the following sections. A general form of the linear programming with interval coefficients can be presented as (Sengupta, Pal and Chakraborty (2001)):

$$\begin{aligned} Min\ z &= \sum_{j=1}^{n}\left[\underline{c}_j,\bar{c}_j\right]x_j \\ s.t. \quad &\sum_{j=1}^{n}\left[\underline{a}_{ij},\bar{a}_{ij}\right]x_j \ge \left[\underline{b}_i,\bar{b}_i\right], \quad for \quad i=1,\dots,m \\ &x_j \ge 0, \quad j=1,\dots,n. \end{aligned} \tag{1}$$

where x_j is a decision variable, $[\underline{c}_j,\bar{c}_j],[\underline{a}_{ij},\bar{a}_{ij}],[\underline{b}_i,\bar{b}_i] \in I(\mathbb{R})$ and $I(\mathbb{R})$ is the set of all interval numbers in $\mathbb{R}$.

According to the operations of interval numbers, each inequality i in (1) can be transformed into 2^{n+1} different extreme inequalities, such as:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \ge b, \tag{2}$$

where $a_j \in \{\underline{a}_{ij},\bar{a}_{ij}\}$ and $b \in \{\underline{b}_i,\bar{b}_i\}, i=1,\dots,m, j=1,\dots,n.$

Consider one inequality i of (1), and let S_k be the set of solutions for the k^{th} extreme inequality version among the 2^{n+1} different extreme inequalities of i.

Now, let

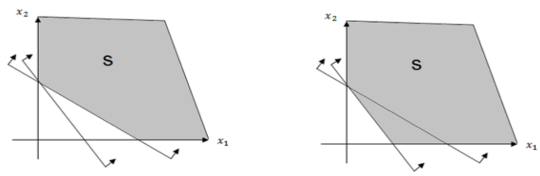
$$\bar{S} = \bigcup_{k=1}^{2^{n+1}} S_k \text{ and } \underline{S} = \bigcap_{k=1}^{2^{n+1}} S_k$$

Figure 1 illustrates how $\bar{S}$ and $\underline{S}$ might appear for an interval inequality having just two possible extreme versions.

Two below definitions are taken from Shaocheng (1994) that will be useful in our discussion.

Definition 3.1. For each constraint inequality i in (1), the inequality $\sum_{j=1}^{n} a_j x_j \ge b$, where $a_j \in [\underline{a}_{ij},\bar{a}_{ij}]$ and $b \in [\underline{b}_i,\bar{b}_i]$ is called the *characteristic formula* for the inequality i in (1).

Definition 3.2. For each constraint inequality in (1), if there exists one characteristic formula such that its solution set is the same as $\bar{S}$ or $\underline{S}$, then this characteristic formula is called the *maximum value range* or *minimum value range* inequality, respectively.



(a) (b)

Fig. 1. The feasible solution set of two different inequalities.
(a)The intersection of the set of solutions of both inequalities,
(b) The union of the set of solutions of both inequalities.

The following theorem shows that how to determine the maximum and minimum value range inequalities for an interval constraint when $x_j \ge 0$. Shaocheng (1994) originally stated this theorems without proof and hence we now give the proof here.

Theorem 3.1. Suppose that we have the interval inequality $\sum_{j=1}^{n}\left[\underline{a}_j,\bar{a}_j\right]x_j \ge \left[\underline{b},\bar{b}\right]$, where $x_j \ge 0$. Then $\sum_{j=1}^{n}\bar{a}_j x_j \ge \underline{b}$ and $\sum_{j=1}^{n}\underline{a}_j x_j \ge \bar{b}$ are respectively the maximum value range and minimum value range inequalities.

Proof: To prove $\sum_{j=1}^{n}\bar{a}_j x_j \ge \underline{b}$ is the maximum value of the inequality interval, we have to show that the result set of this formula is $\bar{S}$. For $\sum_{j=1}^{n}\left[\underline{a}_j,\bar{a}_j\right]x_j \ge \left[\underline{b},\bar{b}\right]$ the characteristic formula of $\sum_{j=1}^{n} a_j x_j \ge b$, where $\sum_{j=1}^{n} a_j x_j \in \left[\sum_{j=1}^{n}\underline{a}_j x_j, \sum_{j=1}^{n}\bar{a}_j x_j\right]$ and $b \in \left[\underline{b},\bar{b}\right]$ for each specific answer is $x = (x_1, x_2, ..., x_n)$. Then for each specific answer of $x_j \geq 0,$ we will have $\sum_{j=1}^{n} a_j x_j \leq \sum_{j=1}^{n} \bar{a}_j x_j$. Also, we have $\underline{b} \leq b \leq \sum_{j=1}^{n} a_j x_j \leq \sum_{j=1}^{n} \bar{a}_j x_j$. Therefore, any answer of the characteristic formula of $\sum_{j=1}^{n} a_j x_j \geq b$ will apply in $\sum_{j=1}^{n} \bar{a}_j x_j \geq \underline{b}$. So $\sum_{j=1}^{n} \bar{a}_j x_j \geq \underline{b}$ is the largest interval value. Similarly, we will have $b \leq \bar{b} \leq \sum_{j=1}^{n} \underline{a}_j x_j \leq \sum_{j=1}^{n} a_j x_j$. As a result, the answers of $\sum_{j=1}^{n} \underline{a}_j x_j \geq \bar{b}$ will apply in the answers of the characteristic formula of $\sum_{j=1}^{n} a_j x_j \geq b$, so $\sum_{j=1}^{n} \underline{a}_j x_j \geq \bar{b}$ is the least interval value, therefore the theorem is proved.

Theorem 3.2. Suppose $Z = \sum_{j=1}^{n} [\underline{c}_j, \bar{c}_j] x_j$ be a given objective function with $x_j \geq 0$, then

$$\sum_{j=1}^{n} \bar{c}_j x_j \geq \sum_{j=1}^{n} \underline{c}_j x_j,$$

for any given vector $x = (x_1, x_2, ..., x_n)$, where $x_j \geq 0,\ j = 1, ..., n.$

Proof: It is obviously that for $x_j \geq 0,$ and using the common interval properties we have

$$\sum_{j=1}^{n} [\underline{c}_j, \bar{c}_j] x_j = [\underline{c}_1, \bar{c}_1] x_1 + [\underline{c}_2, \bar{c}_2] x_2 + ... + [\underline{c}_n, \bar{c}_n] x_n$$

$$= [\underline{c}_1 x_2 + \underline{c}_2 x_2 + ... + \underline{c}_n x_n, \bar{c}_1 x_1 + \bar{c}_2 x_2 + ... + \bar{c}_n x_n]$$

Hence,

$$\bar{c}_1 x_1 + \bar{c}_2 x_2 + ... + \bar{c}_n x_n \geq \underline{c}_1 x_2 + \underline{c}_2 x_2 + ... + \underline{c}_n x_n$$

and then we obtain

$$\sum_{j=1}^{n} \bar{c}_j x_j \geq \sum_{j=1}^{n} \underline{c}_j x_j$$

Definition 3.3. For an interval linear programming problem in minimization form in which $x_j \geq 0,$ the linear function $\sum_{j=1}^{n} \underline{c}_j x_j$ is called the most favourable objective function and also $\sum_{j=1}^{n} \bar{c}_j x_j$ is called the least favourable objective.

Using Theorems 3.1 and 3.2, we will enable to calculate the best and the worst optimized answers to the linear programming problem with interval coefficients. First, we use the most favorable version of the objective function (Theorem 3.2) and the maximum value range inequalities (Theorem 3.1) to determine the best optimal solution, and then we use the least favorable version of the objective function and the minimum value range inequalities to determine the worst optimal solution.

Definition 3.4. Suppose we have the following LPIC problem:

$$Min\ z = \sum_{j=1}^{n} [\underline{c}_j, \bar{c}_j] x_j$$

$$s.t. \quad \sum_{j=1}^{n} [\underline{a}_{ij}, \bar{a}_{ij}] x_j \geq [\underline{b}_i, \bar{b}_i], \ for\ \ i = 1, ..., m \quad (4)$$

$$x_j \geq 0$$

Then the best optimum and worst optimum of the objective function are computed respectively as follows:

$$Min\ \underline{z} = \sum_{j=1}^{n} \underline{c}_j x_j$$

$$s.t. \quad \sum_{j=1}^{n} \bar{a}_{ij} x_j \geq \underline{b}_i, \quad i = 1, ..., m, \quad (5)$$

$$x_j \geq 0, \quad j = 1, ..., n,$$

and

$$Min\ \bar{z} = \sum_{j=1}^{n} \bar{c}_j x_j$$

$$s.t. \quad \sum_{j=1}^{n} \underline{a}_{ij} x_j \geq \bar{b}_i, \quad i = 1, ..., m, \quad (6)$$

$$x_j \geq 0, \quad j = 1, ..., n.$$

The problem (5) and (6), respectively, is called as the optimistic and pessimistic problems. the general form of the optimal value of the objective function of the LPIC problem (4) will be $Z = [\underline{z}, \bar{z}]$, in the other words the optimal value of the problem will be between $\underline{z}$ (the best case scenario) and $\bar{z}$ (the worst case scenario), depending on the coefficient setting of each interval coefficient.

Theorem 3.3. Given an LPIC problem, if for the best optimum we have an infeasible region (i.e. the best optimum does not exist), then the worst optimum has an infeasible region as well, and the original LPIC problem is infeasible.

Proof: Since the region generated by the minimum value range inequalities is a subset of the region generated by the maximum value range inequalities, therefore, the set of minimum value range inequalities is infeasible (i.e. the worst optimum does not exist), if the set of maximum value range inequalities is infeasible.

4. Linear Programming with Random Interval Coefficients

In this section, we present a novel approach on the basis of the Fractile model of stochastic programming for solving linear programming problem involving random interval coefficients. Consider the following linear programming problem:

$$\begin{aligned} &Min\ \ z = c^s x \\ &s.t.\ \ A^s x \geq b^s \\ &\qquad x \geq 0 \end{aligned} \tag{7}$$

where $c^s = (c_1^s, ..., c_n^s)$ so that c_i^s, $i = 1, ..., n$ is a random interval variable, $x = (x_1, ..., x_n)^T$ is an n-dimensional column vector of decision variables, $A^s = \left[a_{ij}^s\right]$ is a $m \times n$ matrix of random interval variables, and $b^s = (b_1^s, ..., b_m^s)^T$ is an m-dimensional column vector of random interval variables. In this paper, we call this problem by LPRIC problem and also suppose all random variables in the upper and lower bound of random intervals are normally distributed.

Considering that the constraints $A^s x \geq b^s$ of the stochastic linear programming problem (7) need not hold almost surely, but they can instead hold with given probabilities, Charnes and Cooper in 1959 initiated the chance-constrained programming. To be more precise, the original *m* constraints

$$\sum_{j=1}^{n} a_{ij}^s x_j \geq b_i^s, \quad i = 1, ..., m \tag{8}$$

Thus, using the concept of chance constrained programming, the constraints of the model of (7) are interpreted as

$$\mathrm{P}\left(\sum_{j=1}^{n} a_{ij}^s x_j \geq b_i^s\right) \geq \beta_i, \ \ i = 1, ..., m \tag{9}$$

where P means probability, and $\beta_1, ..., \beta_m$ are given probabilities of the extents to which constraint violations are admitted. We refer to β_i as the satisficing probability level in this paper. The inequalities (9) are called chance constraints meaning that the i th constraint may be violated, but at most $1-\beta_i$ proportion of the time.

It is obvious that problem (7) is not well-defined due to randomness and intervalness of the coefficients involved in the objective function. In this situation, we cannot optimize the problem likewise deterministic cases.

Here, it is significant to realize that the objective function of problem (7) involves randomness, and it can be regarded as a kind of stochastic programming problems. Furthermore, there are several decision making models such as the expectation optimization model, the variance minimization model, the probability maximization model by Charnes and Cooper (1959) and the fractile optimization model by Geoffrion (1967) for handling stochastic programming problems.

Here, we use an extension of the fractile model of stochastic programming to solve the LPRIC problem which is defined in (7). In this model, a target variable to the objective function is minimized, provided that the probability of the objective function value is smaller than the target variable is guaranteed to be larger than a given assured level.

Considering the constraints in the stochastic linear programming problem (7) as chance constraints introduced in (9), the fractile model minimizing the target variable f under the probabilistic constraints with the assured probability level θ for the objective function and the satisficing probability $\beta_i, i = 1, ..., m$ for the original constraints is formulated as

$$\begin{aligned} &Min \quad f \\ &s.t.\quad \mathrm{P}\left(\sum_{j=1}^{n} c_j^s x_j \leq f\right) \geq \theta \\ &\qquad\quad \mathrm{P}\left(\sum_{j=1}^{n} a_{ij}^s x_j \geq b_i^s\right) \geq \beta_i, \quad i = 1, ..., m, \\ &\qquad\qquad x_j \geq 0,\ \ j = 1, ..., n \end{aligned} \tag{10}$$

where θ is an assured level specified by the decision maker.

Definition 4.1. The decision variable and the target variable of x^* and f^* are respectively said to be an optimal solution and optimistic optimal value, if and only if there does not exist another x , c^s and $f \geq 0$ such that $\mathrm{P}(c^s x \leq f) \geq \theta$ and $f < f^*$. The corresponding values of coefficient $c_i^{*s}, i = 1, 2, ..., n$ are said to be optimistic optimal coefficient.

Definition 4.2 The decision variable and the target variable of x^* and f^* are respectively said to be a optimal solution and pessimistic optimal value, if and only if there does not exist another x and $f \geq 0$ where for all the c^s such that $P(c^s x \le f) \ge \theta$ and $f < f^*$. The corresponding values of coefficient $c_i^{*s}, i = 1,2,...,n$ are said to be pessimistic optimal coefficient.

By using Theorems 3.1 and 3.2, problem (10) will be transformed into the following equivalent problems:

$$\left.\begin{aligned} &Min \quad f \\ &s.t. \quad P\left(\sum_{j=1}^{n} \underline{c}_j^s x_j \le f\right) \ge \theta \qquad (11) \\ &\qquad P\left(\sum_{j=1}^{n} \overline{a}_{ij}^s x_j \ge \underline{b}_i^s\right) \ge \beta_i, \; i = 1,...,m \qquad (12) \\ &\qquad x_j \ge 0, \; j = 1,...,n \end{aligned}\right\} (13)$$

where $\overline{a}_{ij}^s, \underline{c}_j^s, \underline{b}_i^s \in I^S(\mathbb{R})$, and also

$$\left.\begin{aligned} &Min \quad f \\ &s.t. \quad P\left(\sum_{j=1}^{n} \overline{c}_j^s x_j \le f\right) \ge \theta \qquad (14) \\ &\qquad P\left(\sum_{j=1}^{n} \underline{a}_{ij}^s x_j \ge \overline{b}_i^s\right) \ge \beta_i, \; i = 1,...,m \qquad (15) \\ &\qquad x_j \ge 0, \; j = 1,...,n \end{aligned}\right\} (16)$$

where $\underline{a}_{ij}^s, \overline{c}_j^s, \overline{b}_i^s \in I^S(\mathbb{R})$, and $0 \le \theta \le 1, 0 \le \beta_i \le 1, i = 1, ..., m$.

For finding deterministic constraints which are equivalent to the chance constraints, assume that $\underline{b}_i^s$ and $\overline{a}_{ij}^s$ in the chance constraints (12) are random interval variables. Let $m_{\underline{b}_i^s}$ and $\sigma_{\underline{b}_i^s}^2$ be the mean and the variance of $\underline{b}_i^s$, respectively. Also let $m_{\overline{a}_{ij}^s}$ be the mean of $\overline{a}_{ij}^s$ and $V_{\overline{a}_i^s}$ be the variance–covariance matrix of the vector $\overline{a}_i^s = (\overline{a}_{i1}^s, ..., \overline{a}_{in}^s)$. Moreover, assume that $\underline{b}_i^s$ and $\overline{a}_{ij}^s$ are independent of each other.

Since the random variable

$$\frac{\sum_{j=1}^{n} \overline{a}_{ij}^s x_j - \underline{b}_i^s - \left(\sum_{j=1}^{n} m_{\overline{a}_{ij}^s} x_j - m_{\underline{b}_i^s}\right)}{\sqrt{\sigma_{\underline{b}_i^s}^2 + x^T V_{\overline{a}_i^s} x}}, \; i = 1,...,m \qquad (17)$$

is standard normal random variable $N(0,1)$ with mean 0 and variance 1^2, it follows that

$$P\left(\sum_{j=1}^{n} \overline{a}_{ij}^s x_j \ge \underline{b}_i^s\right) = \left(\frac{\sum_{j=1}^{n} \overline{a}_{ij}^s x_j - \underline{b}_i^s - \left(\sum_{j=1}^{n} m_{\overline{a}_{ij}^s} - m_{\underline{b}_i^s}\right)}{\sqrt{\sigma_{\underline{b}_i^s}^2 + x^T V_{\overline{a}_i^s} x}} \ge \frac{-\left(\sum_{j=1}^{n} m_{\overline{a}_{ij}^s} x_j - m_{\underline{b}_i^s}\right)}{\sqrt{\sigma_{\underline{b}_i^s}^2 + x^T V_{\overline{a}_i^s} x}}\right)$$
$$= 1 - \Phi\left(\frac{m_{\underline{b}_i^s} - \sum_{j=1}^{n} m_{\overline{a}_{ij}^s} x_j}{\sqrt{\sigma_{\underline{b}_i^s}^2 + x^T V_{\overline{a}_i^s} x}}\right), \qquad (18)$$

where Φ is the distribution function of the standard normal distribution $N(0,1)$.

Hence, the chance constraints (12) can be transformed into

$$\sum_{j=1}^{n} m_{\overline{a}_{ij}^s} x_j + \Phi^{-1}(1 - \beta_i)\sqrt{\sigma_{\underline{b}_i^s}^2 + x^T V_{\overline{a}_i^s} x} \ge m_{\underline{b}_i^s}, \quad (19)$$

for $i = 1, ..., m$. In the specific case, if for each $i = 1, ..., m$, the parameters $\overline{a}_i^s$ are independent, then the chance constraints (19) can be transformed into

$$\sum_{j=1}^{n} m_{\overline{a}_{ij}^s} x_j + \Phi^{-1}(1 - \beta_i)\sqrt{\sigma_{\underline{b}_i^s}^2 + \sum_{j=1}^{n} Var(\overline{a}_{ij}^s)} \ge m_{\underline{b}_i^s}, \quad (20)$$

Now let $\underline{c}^s = (\underline{c}_1^s, ..., \underline{c}_n^s)$ be a multivariate normal random variable with a mean vector $m_{\underline{c}^s} = (m_{\underline{c}_1^s}, ..., m_{\underline{c}_n^s})$ and an $n \times n$ variance-covariance matrix. Assuming $x \ne 0$, the random variable

$$\frac{\underline{c}^s x - m_{\underline{c}^s} x}{\sqrt{x^T V_{\underline{c}^s} x}} \qquad (21)$$

is standard normal random variable $N(0,1)$. Using (21), it follows that

$$P(\underline{c}^s x \le f) = P\left(\frac{\underline{c}^s x - m_{\underline{c}^s} x}{\sqrt{x^T V_{\underline{c}^s} x}} \le \frac{f - m_{\underline{c}^s} x}{\sqrt{x^T V_{\underline{c}^s} x}}\right) = \Phi\left(\frac{f - m_{\underline{c}^s} x}{\sqrt{x^T V_{\underline{c}^s} x}}\right), \qquad (22)$$

Therefore, the probabilistic constraint

$$\Phi\left(\frac{f - m_{\underline{c}^s} x}{\sqrt{x^T V_{\underline{c}^s} x}}\right) \ge \theta, \qquad (23)$$

where Φ is the distribution function of the standard normal distribution. Let Φ^{-1} be the inverse of Φ, and then (23) is also equivalent to

$$f \geq m_{\underline{c}^s} x + \Phi^{-1}(\theta)\sqrt{x^T V_{\underline{c}^s} x} \qquad (24)$$

Since minimizing f is equivalent to minimizing the right-hand side of (24), the fractile model with the chance constraints (13) can be equivalently transformed to

$$\begin{aligned} Min \quad & \sum_{j=1}^{n} m_{\underline{c}_j^s} x_j + \Phi^{-1}(\theta)\sqrt{x^T V_{\underline{c}^s} x} \\ s.t. \quad & \sum_{j=1}^{n} m_{\bar{a}_{ij}^s} x_j + \Phi^{-1}(1-\beta_i)\sqrt{\sigma_{\underline{b}_i^s}^2 + x^T V_{\bar{a}_i^s} x} \geq m_{\underline{b}_i^s} \qquad (25) \\ & x_j \geq 0, j = 1,\dots,n. \end{aligned}$$

Similarly, with a similar transformation, we have

$$\begin{aligned} Min \quad & \sum_{j=1}^{n} m_{\bar{c}_j^s} x_j + \Phi^{-1}(\theta)\sqrt{x^T V_{\bar{c}^s} x} \\ s.t. \quad & \sum_{j=1}^{n} m_{\underline{a}_{ij}^s} x_j + \Phi^{-1}(1-\beta_i)\sqrt{\sigma_{\bar{b}_i^s}^2 + x^T V_{\underline{a}_i^s} x} \geq m_{\bar{b}_i^s} \qquad (26) \\ & x_j \geq 0, j = 1,\dots,n. \end{aligned}$$

5. Numerical example

Consider the following problem with random interval variable coefficients, where the coefficients of the left-hand sides and the right-hand are independent random variables with the normal distribution:

$$\begin{aligned} Min\ z = & \left[\underline{c}_1^s, \bar{c}_1^s\right] x_1 + \left[\underline{c}_2^s, \bar{c}_2^s\right] x_2 \\ s.t.\ & \left[\underline{a}_{11}^s, \bar{a}_{11}^s\right] x_1 + \left[\underline{a}_{12}^s, \bar{a}_{12}^s\right] x_2 \geq \left[\underline{b}_1^s, \bar{b}_1^s\right] \\ & \left[\underline{a}_{21}^s, \bar{a}_{21}^s\right] x_1 + \left[\underline{a}_{22}^s, \bar{a}_{22}^s\right] x_2 \geq \left[\underline{b}_2^s, \bar{b}_2^s\right] \qquad (27) \\ & \left[\underline{a}_{31}^s, \bar{a}_{31}^s\right] x_1 + \left[\underline{a}_{32}^s, \bar{a}_{32}^s\right] x_2 \geq \left[\underline{b}_3^s, \bar{b}_3^s\right] \\ & x_1 \geq 0 \ , \ x_2 \geq 0 \end{aligned}$$

For the above normal random interval variables, assume that

$\underline{b}_1^s \sim N(4, 2^2)$, $\bar{b}_1^s \sim N(5, 3^2)$,
$\underline{b}_2^s \sim N(16, 1^2)$, $\bar{b}_2^s \sim N(17, 2^2)$,
$\underline{b}_3^s \sim N(8, 0.5^2)$, $\bar{b}_3^s \sim N(12, 2^2)$,

and

Table 1
The expectations and variances of the objective coefficients

	$\underline{c}_1^s$	$\underline{c}_2^s$	$\bar{c}_1^s$	$\bar{c}_2^s$
$E(.)$	3.5	5	4	5.5
$Var(.)$	1	1	1	2

also

Table 2
The expectations and variances of the technical coefficients

	$\underline{a}_{11}^s$	$\underline{a}_{12}^s$	$\underline{a}_{21}^s$	$\underline{a}_{22}^s$	$\underline{a}_{31}^s$	$\underline{a}_{32}^s$	$\bar{a}_{11}^s$	$\bar{a}_{12}^s$	$\bar{a}_{21}^s$	$\bar{a}_{22}^s$	$\bar{a}_{31}^s$	$\bar{a}_{32}^s$
$E(.)$	3	−2	4	16	5	2	4	−1.5	5	17	6	3
$Var(.)$	0.5	1	0.5	0.5	1	0.25	0.5	1	1	0.5	0.5	0.25

Assume that the Decision Maker (DM) specifies the satisficing probability levels as $\beta_i = 0.8,\ i = 1,2,3$ and provides the assured probability level for the probabilistic constraint with respect to the objective function as $\theta = 0.8$.

Problem (27) by using Definition 3.3 be transformed into the following equivalent problems

$$\begin{aligned} Min\ \underline{z} = & \underline{c}_1^s x_1 + \underline{c}_2^s x_2 \\ s.t.\ & \bar{a}_{11}^s x_1 + \bar{a}_{12}^s x_2 \geq \underline{b}_1^s \\ & \bar{a}_{21}^s x_1 + \bar{a}_{22}^s x_2 \geq \underline{b}_2^s \qquad (28) \\ & \bar{a}_{31}^s x_1 + \bar{a}_{32}^s x_2 \geq \underline{b}_3^s \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

and also,

$$\begin{aligned} Min\ \bar{z} = & \bar{c}_1^s x_1 + \bar{c}_2^s x_2 \\ s.t.\ & \underline{a}_{11}^s x_1 + \underline{a}_{12}^s x_2 \geq \bar{b}_1^s \\ & \underline{a}_{21}^s x_1 + \underline{a}_{22}^s x_2 \geq \bar{b}_2^s \qquad (29) \\ & \underline{a}_{31}^s x_1 + \underline{a}_{32}^s x_2 \geq \bar{b}_3^s \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Now by using the Fractile model with the chance constraints, the problem (28) can be transformed into the following equivalent problem:

$$\begin{aligned} Min\ \underline{z} = & 3.5x_1 + 5x_2 + \Phi^{-1}(0.8)\sqrt{x_1^2 + x_2^2} \\ s.t.\ & 4x_1 - 1.5x_2 + \Phi^{-1}(0.2)\sqrt{2^2 + 0.5x_1^2 + x_2^2} \geq 4 \\ & 5x_1 + 17x_2 + \Phi^{-1}(0.2)\sqrt{1^2 + x_1^2 + 0.5x_2^2} \geq 16 \qquad (30) \\ & 6x_1 + 3x_2 + \Phi^{-1}(0.2)\sqrt{0.5^2 + 0.5x_1^2 + 0.25x_2^2} \geq 8 \\ & x_1 \geq 0 \ ,\ x_2 \geq 0 \end{aligned}$$

Then, by using the software LINGO (ver. 11.0), we obtain the optimistic optimal solution and the optimistic value of the objective function as follows:

$$(x_1^*, x_2^*) = (1.711782, 0.5386476)$$
$$\underline{z}^* = 10.20983$$

Also, again by using the Fractile model with the chance constraints, problems (29) be transformed into the following equivalent problem:

$$\begin{aligned} Min\ \bar{z} = & 4x_1 + 5.5x_2 + \Phi^{-1}(0.8)\sqrt{x_1^2 + 2x_2^2} \\ s.t.\ & 3x_1 - 2x_2 + \Phi^{-1}(0.2)\sqrt{3^2 + 0.5x_1^2 + x_2^2} \geq 5 \\ & 4x_1 + 16x_2 + \Phi^{-1}(0.2)\sqrt{2^2 + 0.5x_1^2 + 0.5x_2^2} \geq 17 \qquad (31) \\ & 5x_1 + 2x_2 + \Phi^{-1}(0.2)\sqrt{2^2 + x_1^2 + 0.25x_2^2} \geq 12 \\ & x_1 \geq 0 \ ,\ x_2 \geq 0 \end{aligned}$$

Also, by using software LINGO (ver. 11.0), we obtain the pessimistic optimal solution and the pessimistic optimal value of the objective function as follows:

$$(x_1^*, x_2^*) = (3.027234, 0.4622879)$$
$$\bar{z}^* = 17.28399$$

As a result, the general solution of the problem (27) as follows:

$$z = [\underline{z}, \bar{z}] = [10.20983, 17.28399].$$

The other optimistic and pessimistic optimal solutions and the associated optimistic and pessimistic optimal values of the objective function is given in Table 3.

6. Conclusions

In this presented paper, we have considered linear programming problem with random interval coefficients. We have used an extension of fractile model of stochastic programming for solving it. we saw the mentioned approach is so practical to the real situations. In the proposed approach, we reduced the main model to two sub-problem for determining the optimistic and pessimistic optimal solution. In particular, we solved a numerical example to show the fractile model can prepare a solving process. The main discussion based on the taken results in numerical example part is as follows:

- The best solution for problem (27) is occurred, based on Fractile model in $\beta = 0.1$ and $\theta = 0.1$.
- Generally $\theta \in \left(\frac{1}{2}, 1\right)$ is a constant that determined by the decision maker. From the stochastic point of view, if the value of θ was lower than $\frac{1}{2}$, it means the given constraint would occur with the lower probability which is not a suitable solution for the problem.
- In the discussed problem, two optimistic and pessimistic solutions were gained. The optimistic solution is achieved by considering the lower bound parameters available in problem constraints and as a result of the development of the feasible region.
- This solution would give a better optimum value to interval programming problem because it is chosen in a wider range in compare with the pessimistic solution.
- In stochastic interval programming, the result will ensure with the percentage of confidence because the parameters available in lower and upper bound of problem are random.

Tableb 3
The optimistic and pessimistic solutions for all assured probability levels

	(θ_i, β_j)	$\underline{x}^* = (\underline{x}_1^*, \underline{x}_2^*)$	$\overline{x}^* = (\overline{x}_1^*, \overline{x}_2^*)$	$z^* = [\underline{z}^*, \overline{z}^*]$
$i = 1, j = 1$	(0.1,0.1)	(0.861227,0.582958)	(1.548590,0.489410)	[4.585714,6.697989]
$i = 1, j = 2$	(0.1,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[4.789020,7.303853]
$i = 1, j = 3$	(0.1,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[4.942629,7.773579]
$i = 1, j = 4$	(0.1,0.4)	(1.080462,0.599961)	(2.046477,0.510340)	[5.187163,8.193457]
$i = 1, j = 5$	(0.1,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[5.436362,8.622131]
$i = 1, j = 6$	(0.1,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[5.699145,9.079152]
$i = 1, j = 7$	(0.1,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[5.991198,9.733705]
$i = 1, j = 8$	(0.1,0.8)	(1.711782,0.538648)	(3.027234,0.462288)	[6.369531,10.65636]
$i = 1, j = 9$	(0.1,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[6.963086,12.11666]
$i = 2, j = 1$	(0.2,0.1)	(0.861226,0.582958)	(1.548590,0.489410)	[5.045103,7.444327]
$i = 2, j = 2$	(0.2,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[5.268921,8.132909]
$i = 2, j = 3$	(0.2,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[5.439704,8.668415]
$i = 2, j = 4$	(0.2,0.4)	(1.080462,0.599961)	(2.046477,0.510340)	[5.730942,9.148263]
$i = 2, j = 5$	(0.2,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[6.030684,9.639260]
$i = 2, j = 6$	(0.2,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[6.348175,10.16389]
$i = 2, j = 7$	(0.2,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[6.701587,10.93039]
$i = 2, j = 8$	(0.2,0.8)	(1.711782,0.538648)	3.027234,0.462288()	[7.159125,12.01905]
$i = 2, j = 9$	(0.2,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[7.874848,13.74491]
$i = 3, j = 1$	(0.3,0.1)	(0.861227,0.582958)	(1.548590,0.489410)	[5.377897,7.987119]
$i = 3, j = 2$	(0.3,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[5.617939,8.735859]
$i = 3, j = 3$	(0.3,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[5.801214,9.319220]
$i = 3, j = 4$	(0.3,0.4)	(1.080462,0.599961)	(2.046477,0.510340)	[6.126417,9.842667]
$i = 3, j = 5$	(0.3,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[6.462919,10.37899]
$i = 3, j = 6$	(0.3,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[6.820196,10.95278]
$i = 3, j = 7$	(0.3,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[7.218233,11.80071]
$i = 3, j = 8$	(0.3,0.8)	(1.711782,0.538648)	(3.027234,0.462288)	[7.733374,13.01010]
$i = 3, j = 9$	(0.3,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[8.537947,14.92909]
$i = 4, j = 1$	(0.4,0.1)	(0.861227,0.582959)	(1.548590,0.489410)	[5.377897,8.445100]
$i = 4, j = 2$	(0.4,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[5.912424,9.244598]

$i = 4, j = 3$	(0.4,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[6.106237,9.868309]
$i = 4, j = 4$	(0.4,0.4)	(1.080462,0.599961)	(2.046477,0.510330)	[6.460100,10.42857]
$i = 4, j = 5$	(0.4,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[6.827617,11.00314]
$i = 4, j = 6$	(0.4,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[7.218465,11.61842]
$i = 4, j = 7$	(0.4,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[7.654153,12.53504]
$i = 4, j = 8$	(0.4,0.8)	(1.711782,0.538648)	(3.027234,0.462288)	[8.217898,13.84629]
$i = 4, j = 9$	(0.4,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[9.097437,15.92824]
$i = 5, j = 1$	(0.5,0.1)	(0.861227,0.582958)	(1.548590,0.489410)	[5.929084,8.886118]
$i = 5, j = 2$	(0.5,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[6.196001,9.734495]
$i = 5, j = 3$	(0.5,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[6.399963,10.39708]
$i = 5, j = 4$	(0.5,0.4)	(1.080462,0.599961)	(2.046477,0.510340)	[6.781424,10.99277]
$i = 5, j = 5$	(0.5,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[7.178808,11.60417]
$i = 5, j = 6$	(0.5,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[7.601983,12.25939]
$i = 5, j = 7$	(0.5,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[8.073928,13.24218]
$i = 5, j = 8$	(0.5,0.8)	(1.711782,0.538648)	(3.027234,0.462288)	[8.684475,14.65152]
$i = 5, j = 9$	(0.5,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[9.636205,16.89039]
$i = 6, j = 1$	(0.6,0.1)	(0.861227,0.582958)	(1.548590,0.489410)	[6.199478,9.327236]
$i = 6, j = 2$	(0.6,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[6.479579,10.22439]
$i = 6, j = 3$	(0.6,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[6.693690,10.92584]
$i = 6, j = 4$	(0.6,0.4)	(1.080462,0.599961)	(2.046477,0.510340)	[7.102747,11.55698]
$i = 6, j = 5$	(0.6,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[7.529999,12.20520]
$i = 6, j = 6$	(0.6,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[7.985501,12.90037]
$i = 6, j = 7$	(0.6,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[8.493703,13.94931]
$i = 6, j = 8$	(0.6,0.8)	(1.711782,0.538648)	(3.027234,0.462288)	[9.151053,15.45675]
$i = 6, j = 9$	(0.6,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[10.17497,17.85254]
$i = 7, j = 1$	(0.7,0.1)	(0.861227,0.582958)	(1.548590,0.489410)	[6.480272,9.785117]
$i = 7, j = 2$	(0.7,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[6.774063,10.73313]
$i = 7, j = 3$	(0.7,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[6.998713,11.47495]
$i = 7, j = 4$	(0.7,0.4)	(1.080462,0.599961)	(2.046477,0.510340)	[7.436430,12.14288]
$i = 7, j = 5$	(0.7,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[7.894697,12.82934]
$i = 7, j = 6$	(0.7,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[8.383769,13.56601]
$i = 7, j = 7$	(0.7,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[8.929623,14.68364]
$i = 7, j = 8$	(0.7,0.8)	(1.711782,0.538648)	(3.027234,0.462288)	[9.635577,16.29294]
$i = 7, j = 9$	(0.7,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[10.73446,18.85169]
$i = 8, j = 1$	(0.8,0.1)	(0.861227,0.582958)	(1.548590,0.489410)	[6.813065,10.32791]
$i = 8, j = 2$	(0.8,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[7.123081,11.33608]
$i = 8, j = 3$	(0.8,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[7.360222,12.12574]
$i = 8, j = 4$	(0.8,0.4)	(1.080462,0.599961)	(2.046477,0.510340)	[7.831905,12.83729]
$i = 8, j = 5$	(0.8,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[8.326932,13.56907]
$i = 8, j = 6$	(0.8,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[8.855791,14.35490]
$i = 8, j = 7$	(0.8,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[9.446270,15.55396]
$i = 8, j = 8$	(0.8,0.8)	(1.711782,0.538647)	(3.027234,0.462288)	[10.20983,17.28399]
$i = 8, j = 9$	(0.8,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[11.39756,20.03587]
$i = 9, j = 1$	(0.9,0.1)	(0.861227,0.582958)	(1.548590,0.489410)	[7.270654,11.07425]
$i = 9, j = 2$	(0.9,0.2)	(0.908814,0.603030)	(1.746777,0.499525)	[7.602982,12.16514]
$i = 9, j = 3$	(0.9,0.3)	(0.945683,0.618015)	(1.903755,0.505829)	[7.857298,13.02057]
$i = 9, j = 4$	(0.9,0.4)	(1.080462,0.599961)	(2.046477,0.510340)	[8.375684,13.79209]
$i = 9, j = 5$	(0.9,0.5)	(1.218543,0.582781)	(2.194444,0.513889)	[8.921254,14.58620]
$i = 9, j = 6$	(0.9,0.6)	(1.361636,0.567251)	(2.354584,0.516556)	[9.504821,15.43964]
$i = 9, j = 7$	(0.9,0.7)	(1.516868,0.552978)	(2.627426,0.496813)	[10.15666,16.75065]
$i = 9, j = 8$	(0.9,0.8)	(1.711782,0.538648)	(3.027234,0.462288)	[10.99942,18.64668]
$i = 9, j = 9$	(0.9,0.9)	(2.004884,0.523822)	(3.654075,0.413471)	[12.30932,21.66412]

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